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RELIABLE CONTROL OF DECENTRALIZED SYSTEMS: AN ARE-BASED H-INFINITY APPROACH

Robert Joseph Veillette

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RELIABLE CONTROL OF DECENTRALIZED SYSTEMS: AN ARE-BASED H-INFINITY APPROACH

BY

ROBERT JOSEPH VEILLETTE

B.S., Virginia Polytechnic Institute & State University, 1982
M.S. Clemson University, 1985

THESIS

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ABSTRACT

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The derived decentralized control law consists of a full-order observer of the plant in each control channel. Each observer includes estimates of the controls generated by the other channels and of plant disturbance inputs, based on its own estimate of the state of the plant. All of the observer gains are computed from the solution of a single Riccati-like algebraic equation, while feedback gains are computed from a state-feedback design ARE. The existence of appropriate solutions to the design equations is sufficient to guarantee the various properties of the closed-loop system.

A convexity property of a certain matrix Riccati function allows parameterization of families of control laws with the same desired properties. Each value of the parameter results in controller of the same order as the plant.

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Chapter 1

Introduction

1.1 The H_∞ Control Problem

The so-called H_∞ criterion for control system design has grown popular since it was first introduced by Zames in [37]. Defined for a stable transfer-function matrix $T(s)$ as

$$\|T\|_\infty = \sup_{\omega \in R} \sigma_{\max}\{T(j\omega)\} \tag{1.1}$$

(where $\sigma_{\max}\{\cdot\}$ denotes the maximum singular value), or equivalently as

$$\|T\|_\infty = \sup_{\|w\|_2 = 1} \|z\|_2 \tag{1.2}$$

(where $z(s) = T(s)w(s)$, and $\|\cdot\|_2$ denotes the root-integral-square norm in either the time domain or the frequency domain), the H_∞ norm represents a worst-case cost when the objective is to keep $\|z\|_2$ as small as possible.

The formulation of the standard H_∞ control problem for the linear, time-invariant case generally includes a plant of the form

$$\dot{x} = Ax + Bu + Gw_0, \tag{1.3a}$$

$$y = Cx + w, \tag{1.3b}$$

$$z = \begin{pmatrix} Hx \\ u \end{pmatrix}, \tag{1.3c}$$

where x is the state of the plant, u is the control input, y is a measured output, z is an output to be regulated, and w_0 and w are square-integrable disturbances. The problem is to design a feedback

controller which uses the measurement y to produce a control input u such that, when the loop is closed, the transfer-function matrix from $w_e = \binom{w_0}{w}$ to z has a small H_∞ norm. This H_∞ norm represents a worst-case disturbance attenuation for the closed-loop system. Figure 1.1 depicts the problem setup. (Figures are grouped at the end of the chapter.)

Many familiar problems, in addition to the disturbance-rejection problem, can be recast as a version of this standard problem. Figure 1.2 shows how a frequency-weighted disturbance-rejection problem, a model-reference problem, and a tracking problem can be transformed into the standard form of Figure 1.1. In each case, any exogenous input is included in the disturbance vector w, and the regulated output vector z is the error to be minimized. Note that for the tracking problem, and for the model-reference problem when $M(s)$ is not strictly proper, z must be allowed to depend directly on the exogenous input. While this is not the case for the plant (1.3), this generalization can be accommodated; see [15].

Closely related to the H_∞ disturbance-rejection problem is that of robust stabilization of a plant $P_0(s) + \Delta(s)$, where the nominal plant $P_0(s)$ is known, but $\Delta(s)$ is restricted only by a bound on $\|\Delta\|_\infty$. The controller which solves this robust stabilization problem with the largest admissible bound on $\|\Delta\|_\infty$ is that which provides the smallest possible H_∞ norm for the nominal closed-loop system [38]. See [12] and its references for a survey of the literature up to 1987 on robust stability and H_∞ performance.

Until quite recently, the computations for designing an H_∞-optimal controller (summarized in [23]) were formidable: They included stable coprime factorizations of the plant and a stabilizing controller, plus a parameterization of all stabilizing controllers [36], leading to an equivalent model-matching problem; inner-outer and spectral factorizations, leading to an equivalent H_∞-L_∞ approximation problem [24]; and solution of the H_∞-L_∞ approximation problem by Hankel-norm approximation methods [16].

Developments in the last few years, however, have simplified H_∞ control design considerably. Results such as those in [19], [22], and [25] have established that robustly stabilizing control designs can be computed from algebraic Riccati equation (ARE) solutions. Still more recently, [14], [17], [21], and [39] have given H_∞ disturbance-rejection designs also computed from ARE solutions, and [3] has given an ARE-based design which simultaneously gives H_∞ and LQG cost bounds for the closed-loop system.

Given any control input u to the plant (1.3), define the cost functional

$$J(u) = \sup \left\{ \frac{\|z\|_2}{\|w_0\|_2} : w_0 \in L_2[0,\infty), \quad w_0 \neq 0 \right\}.$$

Note that the measurement (1.3b), and hence the measurement noise w, is not considered in this definition. Therefore, the cost $J(u)$ is associated with open-loop controls or state-feedback controls. In the case where u is a state-feedback control, $J(u)$ is the H_∞ norm of the closed-loop transfer-function matrix from w_0 to z. Define the optimal cost as

$$\alpha_\infty = \inf \{J(u) : u \in L_2[0,\infty)\}. \tag{1.4}$$

The following theorem from [14] gives a means of determining α_∞, and also establishes that there exists a state-feedback control law which achieves any H_∞-norm bound larger than α_∞.

Theorem 1.1. *For the plant (1.3) with (A, B) stabilizable, and (A, H) detectable, the bound*

$$\alpha_\infty < \alpha$$

holds if and only if

$$A^T X + XA + \frac{1}{\alpha^2} XGG^T X - XBB^T X + H^T H = 0, \tag{1.5}$$

with $X \geq 0$ *and* $A_\alpha \equiv A - BB^T X + \alpha^{-2} GG^T X$ *Hurwitz. If so, then use of the state-feedback control law*

$$u = -B^T X x \tag{1.6}$$

stabilizes the plant, and gives a closed-loop transfer-function matrix

$$T(s) = \begin{pmatrix} H \\ -B^T X \end{pmatrix} (sI - A + BB^T X)^{-1} G$$

from w_0 *to* z *satisfying* $\|T\|_\infty < \alpha$.

Note that there may be several solutions $X \geq 0$ to the ARE (1.5). The condition that A_α must be Hurwitz specifies a particular solution $X \geq 0$, and also assures that the Hamiltonian matrix associated with the ARE (1.5) has no $j\omega$-axis eigenvalues.

Theorem 1.1 establishes a relationship between H_∞-optimal control and LQ-optimal control. The state-feedback control law given by (1.5) and (1.6) becomes H_∞-optimal as α approaches the

lower bound α_∞, reflecting a great concern with disturbance rejection, along with confidence in the disturbance "directions" characterized by the matrix G; on the other hand, as α approaches ∞, the design becomes LQ-optimal, reflecting little concern with the particular disturbance w_0. Of course, choosing α very large in a design does not imply that the closed-loop system will have a very large H_∞ norm; rather, the bound $\|T\|_\infty < \alpha$ becomes very conservative for $\alpha \gg \alpha_\infty$. An easy method for computing a tighter H_∞-norm bound for the design of Theorem 1.1, as well as for various ARE-based state-feedback designs, is developed in [28]. Applied to examples, this method shows that even the LQ-optimal design corresponding to $\alpha = \infty$ above often has acceptable disturbance-attenuation properties. The results of [28] and the results of this thesis share the same ARE-based approach to computing or guaranteeing H_∞-norm bounds; however, the results of [28] are not considered here in detail, since they are essentially analysis tools which provide *a posteriori* bounds for systems designed by several methods. This thesis is concerned with new design methods which provide *a priori* bounds for the closed-loop systems.

If the control u for plant (1.3) must be generated by a controller that uses only the measurement y given by (1.3b), then the greatest lower bound of the set of achievable closed-loop H_∞ norms is generically greater than α_∞ defined in (1.4). The following theorem from [14,13] gives a means of determining this greatest lower bound, and also gives an output-feedback control law which guarantees any given H_∞-norm bound achievable by output feedback.

Theorem 1.2. *In the plant (1.3), assume (A,B) stabilizable, (A,C) detectable, (A,G) stabilizable, and (A,H) detectable. Then there exists a stabilizing controller such that the closed-loop transfer-function matrix $T(s)$ from w_e to z satisfies $\|T\|_\infty < \alpha$ if and only if*

$$A^TX + XA + \frac{1}{\alpha^2}XGG^TX - XBB^TX + H^TH = 0 \tag{1.5}$$

with $X \geq 0$ and $A_\alpha \equiv A - BB^TX + \alpha^{-2}GG^TX$ Hurwitz,

$$AY + YA^T + \frac{1}{\alpha^2}YH^THY - YC^TCY + GG^T = 0 \tag{1.7}$$

with $Y \geq 0$, $A - YC^TC + \alpha^{-2}YH^TH$ Hurwitz, and

$$\sigma_{\max}\{YX\} < \alpha^2. \tag{1.8}$$

If so, then the output-feedback control law

$$\dot{\xi} = \left(A + \frac{1}{\alpha^2}GG^TX - BB^TX - LC\right)\xi + Ly, \tag{1.9a}$$

$$u = -B^T X \xi \tag{1.9b}$$

with

$$L = \left(I - \alpha^{-2} Y X\right)^{-1} Y C^T \tag{1.10}$$

stabilizes the plant and gives a closed-loop transfer-function matrix from w_e to z satisfying $\|T\|_\infty < \alpha$.

The controller (1.9) can be rewritten in the form

$$\dot{\xi} = A\xi + Bu + G\hat{w}_0 + L(y - C\xi), \quad u = -B^T X \xi, \tag{1.11}$$

with

$$\hat{w}_0 = \frac{1}{\alpha^2} G^T X \xi. \tag{1.12}$$

The form (1.11) is essentially that of an observer: It mimics the plant dynamics (1.3a) and includes a correction term based on the measurement y. Unlike a standard observer, it includes a term representing an unmeasurable disturbance. Equation (1.12) corresponds to a state-feedback model of the disturbance w_0. Like an LQG-optimal control law, then, the H_∞ control of Theorem 1.2 has an observer structure, with the observer gains computed from an algebraic Riccati equation. As in an LQG design, the observer design equation (1.7) is a dual form of the state-feedback design equation (1.5). In fact, setting $\alpha = \infty$ reduces (1.5) and (1.7) to LQG design equations.

The difference between the LQG and H_∞ designs can be interpreted as in the state-feedback case: The H_∞ design is more concerned with disturbance rejection. The form (1.12) represents an approximation of the disturbance $w_{0,\text{worst}} = \frac{1}{\alpha^2} G^T X x$, which is given in [33] as a worst disturbance in a game setting where the state-feedback control u plays against the disturbance w_0 and initial conditions.

The set of all stabilizing output-feedback controllers guaranteeing a particular H_∞-norm bound is given in [13] as follows:

Theorem 1.3. *If the conditions of Theorem 1.2 are satisfied, then the set of all finite-dimensional stabilizing output-feedback controllers guaranteeing $\|T\|_\infty < \alpha$ is given by*

$$\dot{\xi} = \left(A + \frac{1}{\alpha^2} G G^T X - B B^T X - LC\right)\xi + Ly + \left(I - \alpha^{-2} Y X\right)^{-1} B v \tag{1.13a}$$

$$u = -B^T X \xi + v, \tag{1.13b}$$

where L is given by (1.10), and v is given in the frequency domain by $v(s) = Q(s)(y(s) - C\xi(s))$ with $Q(s)$ being any finite-dimensional, stable transfer-function matrix satisfying $\|Q\|_\infty < \alpha$.

Note that picking $Q(s) = 0$ results in the central control law of Theorem 1.2. It is possible that some controller given by Theorem 1.3 may be of lower order than the plant, but there is no clear way of choosing $Q(s)$ to be sure to obtain such a lower-order controller. Further, many choices of $Q(s)$ will yield controllers of order higher than the plant.

1.2 The Decentralized Case

The H_∞ control problem can be generalized to the decentralized case, where the plant is described by

$$\dot{x} = Ax + \sum_{i=1}^{q} B_i u_i + G w_0, \tag{1.14a}$$

$$y_i = C_i x + w_i, \quad i \in \{1, 2, \ldots, q\}, \tag{1.14b}$$

$$z = \begin{pmatrix} Hx \\ u_1 \\ \vdots \\ u_q \end{pmatrix}. \tag{1.14c}$$

The decentralized control structure consists of the restriction that each control input u_i must be generated by an independent controller which uses only the corresponding measurement y_i. This restriction corresponds to the practical need in some situations to control a large system by means of several controllers which, because of physical separation or other reasons, cannot exchange measurement or control information. The goal is to design dynamic controllers

$$\dot{\xi} = A_{di}\xi_i + B_{di}y_i, \quad i \in \{1, 2, \ldots, q\}, \tag{1.15a}$$

$$u_i = C_{di}\xi_i, \quad i \in \{1, 2, \ldots, q\}, \tag{1.15b}$$

which will stabilize the decentralized plant (1.14), and provide a predetermined H_∞-norm bound for the closed-loop system.

Many past results in decentralized control are concerned with conditions for existence of stabilizing decentralized control laws. Davison reviews some of these results in [9]. One important

result [32] is based on the concept of the "fixed modes" of a decentralized plant. A decentralized fixed mode of plant (1.14) is defined as any eigenvalue of A which cannot be moved by a static decentralized feedback; that is, the set of fixed modes is defined as

$$\bigcap_{K \in D} \Lambda\{A + BKC\},$$

where D is the set of block-diagonal matrices whose blocks are sized to conform in the obvious way with the sizes of the blocks of B and C defined by

$$B = (B_1 \; B_2 \; \ldots \; B_q),$$

$$C^T = \left(C_1^T \; C_2^T \; \ldots \; C_q^T\right),$$

and $\Lambda\{\cdot\}$ denotes the set of eigenvalues. The main result of [32] is that a plant can be stabilized by a linear, time-invariant decentralized control law if and only if it has no fixed modes in the closed right-half plane. Hence, the concept of a (decentralized) fixed mode is an extension of the concept of an uncontrollable or unobservable mode in the centralized case. An algebraic condition given in [1] equivalent with the presence of fixed modes in a strictly-proper decentralized plant is that, for some renumbering of control channels, for some integer $t \leq q$, and for some complex number s,

$$\operatorname{rank} \begin{pmatrix} sI - A & B_1 & \ldots & B_t \\ C_{t+1} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ C_q & 0 & \ldots & 0 \end{pmatrix} < n. \tag{1.16}$$

If (1.16) holds, then s is a fixed mode of the plant. The degenerate case $t = 0$ in (1.16) corresponds to a plant which is not observable by all the measurements jointly, while $t = q$ in (1.16) corresponds to a jointly uncontrollable plant.

Some other results give conditions under which a decentralized plant can be made controllable and observable through a single control channel by some feedback applied in the other channels. For example, a main result of [11], restated as Corollary 2 of [5], states that if a controllable two-channel plant is completely observable by one channel, there exists some nondynamic feedback in that channel which will make the plant completely controllable by the other channel. That is, if $(A, [B_1 \; B_2])$ is a controllable pair and (A, C_2) is an observable pair, then there exists a constant

matrix K such that $(A+B_2KC_2, B_1)$ is a controllable pair. Other results (for example, Corollary 3 of [5] and Theorem 1 of [6]) give more general, but more complicated, conditions related to the zeros of the plant's transfer-function matrix and the connectivity of the plant's graph. More recently, [35] gives extensions of the results of [5] and [6] to the multi-channel, non-strictly-proper case. Results in [35] depend on the concept of a strongly-connected plant. A decentralized plant with the open-loop transfer-function matrix $P(s)$ given by

$$\begin{pmatrix} y_1(s) \\ \vdots \\ y_q(s) \end{pmatrix} = \begin{pmatrix} P_{11}(s) & \dots & P_{1q}(s) \\ \vdots & & \vdots \\ P_{q1}(s) & \dots & P_{qq}(s) \end{pmatrix} \begin{pmatrix} u_1(s) \\ \vdots \\ u_q(s) \end{pmatrix}$$

is defined to be strongly connected if for every renumbering of control channels, and for every integer t satisfying $0 < t < q$,

$$\begin{pmatrix} P_{1,t+1}(s) & \dots & P_{1q}(s) \\ \vdots & & \vdots \\ P_{t,t+1}(s) & \dots & P_{tq}(s) \end{pmatrix} \neq 0.$$

A main result of [35] is that if a decentralized plant is strongly connected, and if it has no fixed modes, then almost any preapplied nondynamic decentralized feedback makes the plant controllable and observable in each control channel.

In [7] appear necessary and sufficient conditions for existence of a solution to a decentralized robust servomechanism problem. The key point of [7] is that, provided the outputs to be regulated can also be measured, a decentralized control which provides tracking and disturbance rejection for a certain class of reference and disturbance inputs exists for almost all plants. For decentralized plants composed of interconnected subsystems, [7] and [8] establish similar facts: If each subsystem admits a solution to a centralized robust servomechanism problem, then the composite system admits a solution to the decentralized robust servomechanism problem, provided the subsystem interconnections are sufficiently weak or that the subsystems are interconnected only through subsystem inputs and outputs.

There have been a number of approaches to decentralized design. Existence results like those of [5] and [35] suggest a simple one: Preapply to the plant a non-dynamic decentralized feedback – almost any one will do – which will transfer sufficient control authority to one control channel

that the system can be at least stabilized from there. However, there is a great deal of design freedom in this approach, and no clear way of making optimal design choices. The existence results of [7] and [8] also suggest a design approach for decentralized control of composite systems: Design a separate controller for each subsystem, ignoring interconnections. This approach is depicted in Figure 1.3. While there are many good ways of designing the subsystem controllers, there is generally no guarantee that they will stabilize the overall system.

A decentralized design approach is given in [20] for so-called multilevel systems. Multilevel systems, some examples of which are depicted in Figure 1.4, are composite systems whose subsystem interconnections are one-way and loop-free. The approach is to design optimal controllers for the subsystems starting at the top level, optimizing each controller without accounting for the dynamics of lower-level subsystems. The inputs from higher-level subsystems are modelled as disturbance inputs obeying known dynamic equations, since higher-level subsystem controls are designed earlier. Thus, the design method is sequential in nature. The resulting design guarantees stability, plus optimality of each separate subsystem given the designs of higher-level subsystem controllers.

Another sequential design procedure for general decentralized systems appears in [10]. Here, the i^{th} controller is designed to control the plant plus the first $i-1$ controllers. That is, the i^{th} controller design views the first $i-1$ controllers not as sources of disturbance inputs, but as part of the plant to be controlled. The resulting designs guarantee stability, and each controller can be designed to optimize a cost functional which depends only on the plant and previously-designed controllers.

A drawback of sequential designs is that, when dynamic compensators are used, the effective order of the plant to be controlled increases with each controller added. Many controller designs result in controllers the same order as the plant model; thus it is easy to envision a sequential design where the first controller designed has order n, the second has order $2n$, the third has order $4n$, and so on. To get around this problem, some controller reduction may be done at each step so as to keep each controller at some fixed order. Once all the controllers are designed, they may be redesigned, now taking into account the presence of all designed controllers, but retaining the same low order. Hence, such a design is both sequential and iterative, with each controller refined at each iteration. For example, in the iterative sequential design in [2], each controller design takes into account the presence of previously designed controllers, but is obtained from optimal projection equations and

therefore has a fixed (low) order. The design equations correspond to a necessary condition for the decentralized control law to minimize a steady-state cost functional in the presence of random noise inputs, subject to the *a priori* constraints on controller order. If the design equations can be solved, the cost associated with the closed-loop system is guaranteed to decrease with each design iteration.

A nonsequential approach applicable to general decentralized systems [18] is to consider the whole plant in the design of each controller, but to ignore the effects of other controllers. This approach is depicted in Figure 1.5. Under certain restrictions on the strength of plant/controller interconnections, the controllers acting together are guaranteed to stabilize the plant. The restrictions are technical conditions which grow out of a study of stability via generalized overlapping decompositions of the closed-loop system.

1.3 Reliable Control

In a centralized control system, controller failure leaves the plant without control input. This condition is undesirable, especially if the plant is unstable. On the other hand, a single controller failure in a decentralized control system is not necessarily catastrophic, even for an unstable plant. Hence, the use of multiple controllers in a decentralized control design provides the potential for increased system reliability. However, for existing design methods, whether or not a given decentralized system is actually reliable despite possible controller outages depends upon the particular example, and can be determined by analysis only when the design is completed.

In [26], Šiljak considers the approach of designing multiple separate controllers, each of which stablizes the plant by itself. There are technical restrictions on plant/controller interconnections necessary to guarantee that all the controllers together, or some subset of the controllers, also stabilize the system.

In [4], Cho, Bien, and Kim consider the approach of designing a redundant controller to operate in parallel with a baseline stabilizing controller. The redundant controller is a model-reference adaptive controller whose reference model is that of the baseline closed-loop system. For various failure modes, the redundant scheme guarantees stability and asymptotic reference-tracking.

In [31], Vidyasagar establishes some results on the reliable stabilization of a plant by two controllers summed together. For example, given any stabilizing controller for a plant, [31] gives

a means of computing a second stabilizing controller such that the two controllers added together also stabilize the plant. The computation involves stable coprime factorizations of the plant and the first controller, and may result in a second controller of high order.

Related to the issue of reliable control is that of strong stabilization, defined as the stabilization of a plant by an open-loop stable controller. If the input of a strongly stabilizing controller becomes disconnected from the plant output, at least the controller will not apply unwanted large inputs to the plant. The so-called "pole-zero interlacing property" given in [31] is a well-known necessary and sufficient condition for the existence of a strongly stabilizing controller. As with existing reliable designs, strong stability results are mainly concerned that the system simply remain stable despite a controller failure.

1.4 Contribution of This Thesis

This thesis presents a new method for designing decentralized control systems to accomplish a variety of design goals. The basic design method produces a decentralized control law which guarantees closed-loop stability and a worst-case (H_∞) disturbance-attenuation bound when applied to the plant (1.14). The control law consists of a full-order observer in each control channel. Like the centralized controller (1.9), each observer uses an estimate such as that given in (1.12) to approximate the disturbance w_0. In addition, each observer uses estimates of the controls applied in the other channels. These estimates correspond to the strategy shared by the controllers for approximating the state-feedback control law given by (1.5) and (1.6).

The decentralized control law uses the state-feedback gains computed from (1.5) for feedback of the state estimates, and observer gains computed from a single Riccati-like design equation; hence, the controllers are designed all together, rather than sequentially. Since each controller is based on a full-order observer of the plant, each has the same order as the plant. The existence of appropriate solutions to the design equations is sufficient to guarantee the stability and performance of the closed-loop system under the decentralized control law. No other condition on the subsystem interconnections or the plant/controller interconnections is needed. Of course, the design equations will have no solution if the plant has any unstable fixed modes.

Variations on the design equations allow the design of robust, reliable, or strongly-stabilizing decentralized control laws. Each design guarantees stability and a predetermined degree of distur-

bance attenuation. In addition, the robust design can tolerate constant plant uncertainties from a given bounded set; the reliable design can tolerate outages of any or all controllers in a predefined subset of controllers; and the strongly stabilizing design guarantees that all controllers from some predefined subset will be open-loop stable. A centralized reliable design is also developed which can tolerate outages of some sensors or actuators.

The various robust, reliable, and strongly stabilizing designs developed can be combined into a design which will guarantee any or all of the desired properties at once.

Finally, a convexity property of the matrix Riccati function

$$R(X) = F^T X + XF + \frac{1}{\alpha^2} XGG^T X + H^T H, \tag{1.17}$$

with F Hurwitz, allows easy computation of families of matrices $Z \geq 0$ satisfying $R(Z) \leq 0$, and hence, families of decentralized control laws with any of the desired properties. For any control law in the family, the decentralized controllers all have the same order as the plant.

Chapter 2 describes the approach taken to developing new decentralized control laws. The main result of Chapter 2 is Lemma 2.1, which is the basis for all the designs which follow. The result consists of a sufficient condition in the form of an algebraic Riccati inequality

$$R(X) = F^T X + XF + \frac{1}{\alpha^2} XGG^T X + H^T H \leq 0 \tag{1.18}$$

which guarantees the H_∞-norm bound $\|T\|_\infty \leq \alpha$ for the transfer-function matrix

$$T(s) = H(sI - F)^{-1} G.$$

Based on Lemma 2.1, simple new derivations of the control laws of Theorems 1.1 and 1.2 are given. These derivations give insight into the decentralized designs presented in the following chapters.

Chapter 3 presents the basic decentralized control design, which is characterized by closed-loop stability and a predetermined degree of worst-case disturbance attenuation. The basic design results in a closed-loop system which satisfies (1.18) with equality holding. Two examples follow the derivation.

Chapter 4 presents a decentralized design which has the properties of the basic design, and is also robust with respect to structured uncertainty in the plant. The robust design results in a closed-loop system whose matrices satisfy (1.18) for any plant uncertainty in a bounded admissible

set. Following the derivation is a robust state-feedback design example. This example illustrates the concept of robust design, while avoiding the complicating details of decentralized design.

Chapter 5 presents new results on reliable control. Similar to the robust design, the reliable design method results in a closed-loop system which satisfies (1.18) for any admissible controller failures. The design method is first developed in the centralized case, where reliability is guaranteed despite possible outages of some sensors or actuators. Then, a decentralized design method is derived which guaratees reliability with respect to possible controller outages. An example demonstrates that the resulting reliable decentralized design can tolerate controller outages which would cause instability for the basic design. A new design method is also presented which guarantees that the controller, or some prespecified subset of controllers in the decentralized case, will be open-loop stable.

Chapter 6 presents the properties of the matrix Riccati function $R(X)$ which allow computation of families of decentralized control laws with desired disturbance-attenuation, robustness, reliability, or strong stabilization properties. A family of state-feedback designs is presented first. Then it is shown that a family of centralized observers with the desired properties can be computed for each member of the family of state-feedback controls. In the decentralized case, however, only one set of decentralized observer gains can be computed for each state-feedback control, yielding a family of decentralized control laws corresponding exactly with the family of state-feedback control laws.

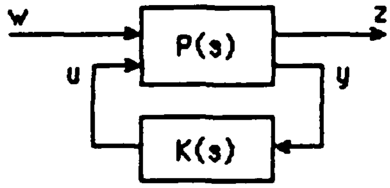
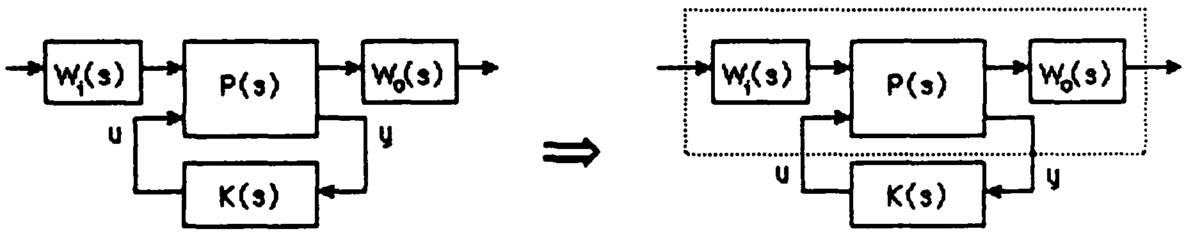
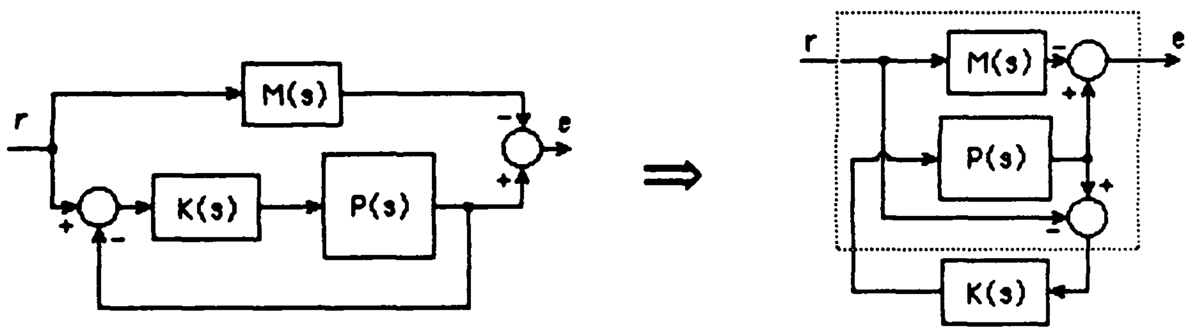


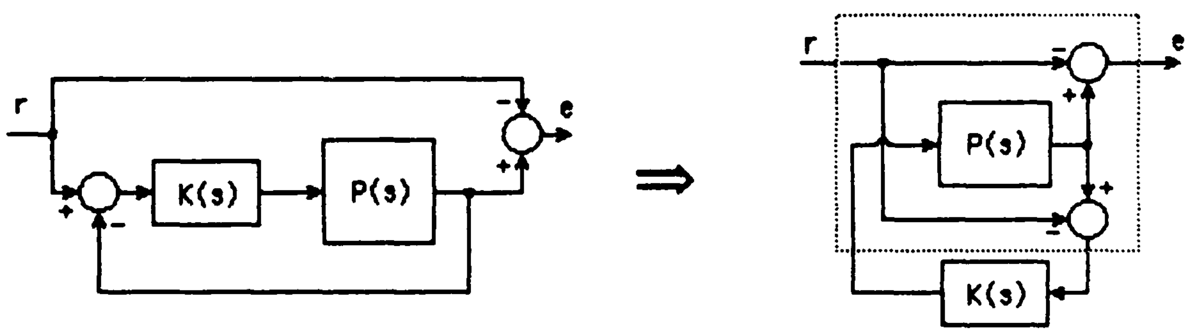
Figure 1.1: The standard H_∞ problem setup.



(a) A frequency-weighted disturbance-rejection problem.



(b) A model-reference problem.



(c) A tracking problem.

Figure 1.2: Reformulation of familiar problems to fit the standard form.

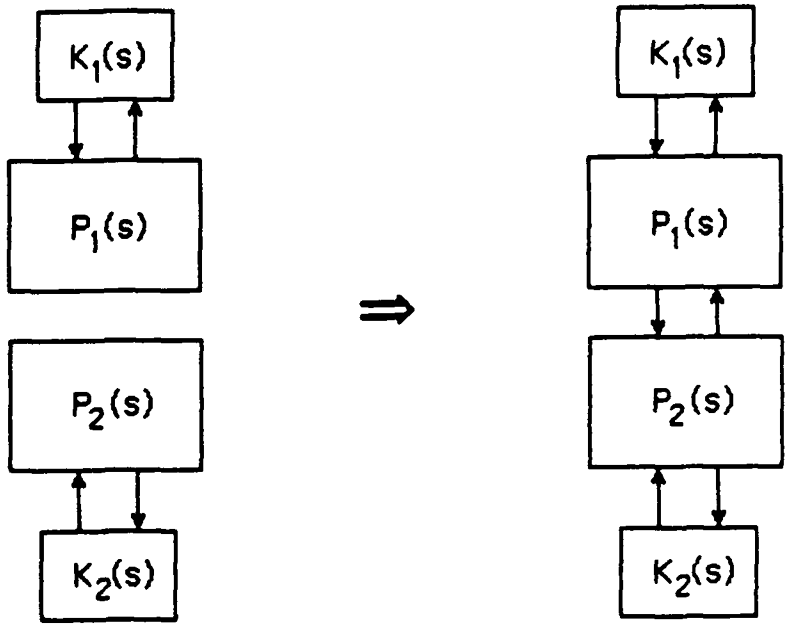


Figure 1.3: Decentralized design ignoring subsystem interconnections.

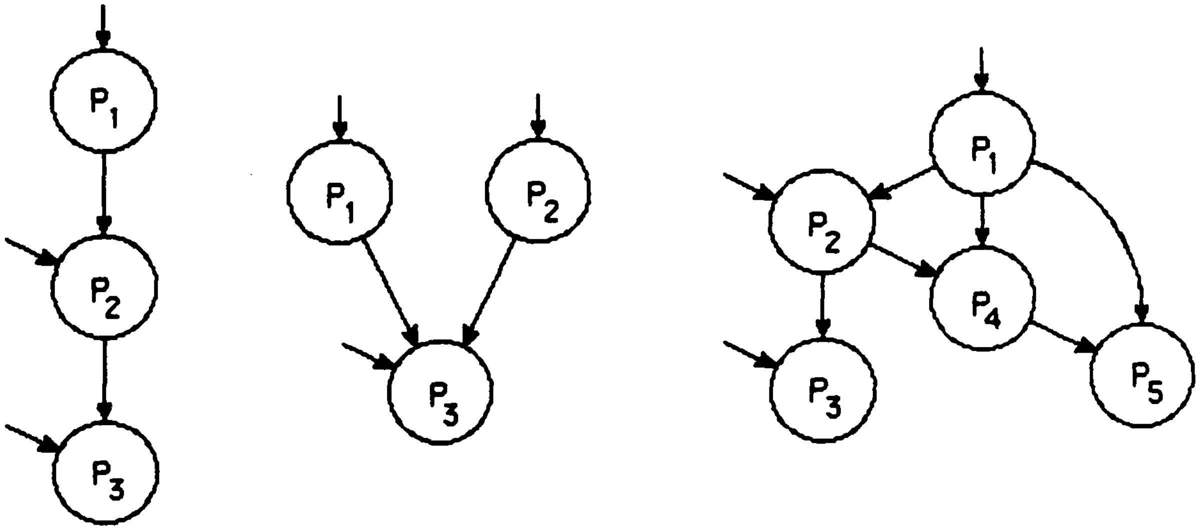


Figure 1.4: Some examples of multilevel systems.

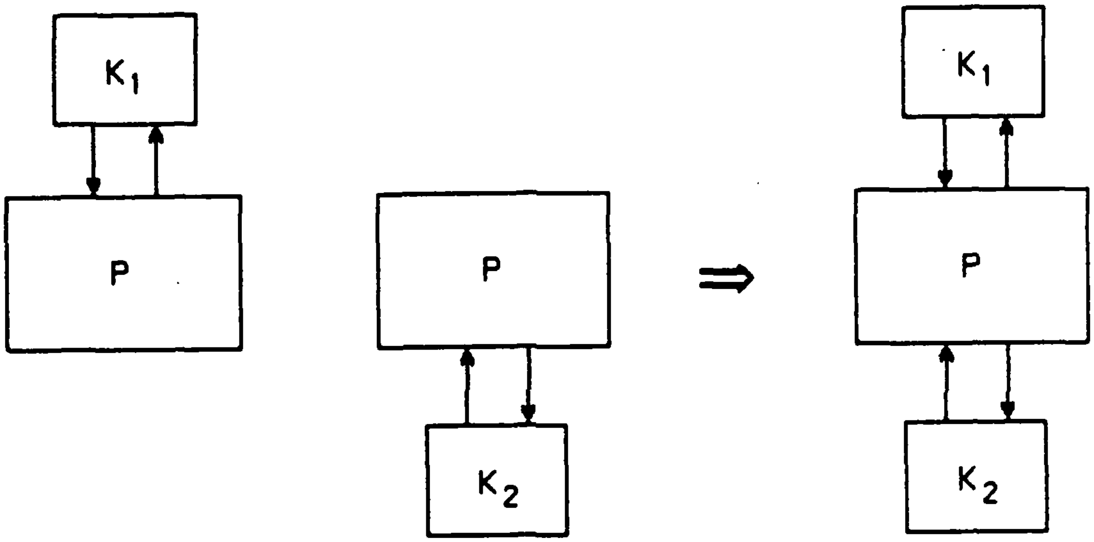


Figure 1.5: Decentralized controller design ignoring other controllers.

Chapter 2

The Approach

2.1 The Key Lemma

The following lemma establishes a sufficient condition, in the form of an "algebraic Riccati inequality," for a given system to be stable and have a particular H_∞-norm bound. The lemma is a simple extension of Lemma 1 of [34].

Lemma 2.1. *Let $T(s) = H(sI - F)^{-1}G$, with (F, H) a detectable pair. If there exist a real matrix $X \geq 0$ and a positive scalar α such that*

$$F^T X + XF + \frac{1}{\alpha^2} XGG^T X + H^T H \leq 0, \tag{2.1}$$

then F is Hurwitz, and $T(s)$ satisfies

$$\|T\|_\infty \leq \alpha. \tag{2.2}$$

Proof: Suppose (2.1) holds, with $X \geq 0$. To show that F is Hurwitz, let $v \neq 0$ satisfy

$$Fv = \lambda v.$$

Multiply (2.1) on the left by v^* and on the right by v to obtain

$$2Re(\lambda)v^* Xv + \frac{1}{\alpha^2} v^* XGG^T Xv + v^* H^T Hv \leq 0. \tag{2.3}$$

Now, $2Re(\lambda)v^* Xv \leq 0$ since all other terms on the left-hand side of (2.3) are non-negative. If $Re(\lambda)v^* Xv < 0$, then $v^* Xv > 0$ and $Re(\lambda) < 0$. If, on the other hand, $Re(\lambda)v^* Xv = 0$, then all terms in (2.3) must be zero. Therefore, the eigenvector v of F is in the null space of H. Since

(F, H) is detectable, the corresponding eigenvalue must be in the open left-half plane. In either case, $Re(\lambda) < 0$; thus, F is Hurwitz.

Now, to prove (2.2), let $\omega \in \mathbb{R}$; add and subtract $j\omega X$ to obtain from (2.1)

$$-(-j\omega I - F^T)X - X(j\omega I - F) + \frac{1}{\alpha^2}XGG^TX + H^TH \leq 0. \tag{2.4}$$

Since F is Hurwitz, $(j\omega I - F)$ is invertible. Define

$$K(j\omega) = \frac{1}{\alpha^2}G^TX(j\omega I - F)^{-1}G;$$

pre-multiply (2.4) by $\frac{1}{\alpha}G^T(-j\omega I - F^T)^{-1}$, and post-multiply by $\frac{1}{\alpha}(j\omega I - F)^{-1}G$ to obtain

$$-K(j\omega) - K^T(-j\omega) + K^T(-j\omega)K(j\omega) + \frac{1}{\alpha^2}T^T(-j\omega)T(j\omega) \leq 0,$$

which gives

$$I - \frac{1}{\alpha^2}T^T(-j\omega)T(j\omega) \geq [I - K^T(-j\omega)][I - K(j\omega)].$$

Therefore, for all $\omega \in \mathbb{R}$,

$$I - \frac{1}{\alpha^2}T^*(j\omega)T(j\omega) \geq [I - K(j\omega)]^*[I - K(j\omega)] \geq 0,$$

which implies (2.2). Q.E.D.

2.2 The General Approach

Lemma 2.1 suggests a particular view of the H_∞ control designs of Theorems 1.1 and 1.2, along with a new approach to decentralized H_∞ design. The approach is to first fix a controller structure, so as to determine the form of the closed-loop system

$$\dot{x}_e = F_ex_e + G_ew_e, \quad z = H_ex_e, \tag{2.5}$$

and then select controller gains so that, for some $P_e \geq 0$, the algebraic Riccati equation

$$F_e^TX_e + X_eF_e + \frac{1}{\alpha^2}X_eG_eG_e^TX_e + H_e^TH_e + P_e = 0 \tag{2.6}$$

has a solution $X_e \geq 0$. By Lemma 2.1, if (F_e, H_e) is a detectable pair, then the closed-loop system (2.5) is stable, and $T(s) = H_e(sI - F_e)^{-1}G_e$ satisfies $\|T\|_\infty \leq \alpha$.

In the designs of Theorems 1.1 and 1.2, controller parameters are chosen to depend on solutions of the design equations (1.5) and (1.7), so that a solution $X_e \geq 0$ of (2.6) is guaranteed to exist, with $P_e = 0$. In Chapter 3, this same approach is applied to decentralized control. To guarantee that (2.6) is satisfied, the observer gains are computed from the solution of a Riccati-like algebraic equation.

Choosing $P_e = 0$ in (2.6) yields what we shall call the "basic design," characterized by stability and the H_∞-norm bound $\|T\|_\infty \leq \alpha$. Choosing some other $P_e \geq 0$ allows greater design freedom to obtain "special designs" with additional desired properties. Specifically, particular choices of $P_e \geq 0$ are used in this thesis to guarantee the following additional properties:

(i) Robustness with respect to constant structured uncertainty in the plant A-matrix, when the uncertainty belongs to a predetermined bounded set;

(ii) Reliability with respect to controller outages, where outages may occur in any or all of the controllers in a predefined subset of controllers;

(iii) Strong stabilization, in the sense that all controllers in a predefined subset of controllers will be guaranteed open-loop stable.

2.3 The Approach Applied to Centralized Control

To provide insight into the approach to decentralized design described above, a new derivation of the control law of Theorem 1.2 is now given, based on Lemma 2.1. This derivation, which also appears in [29], is not a complete proof of Theorem 1.2, in that it establishes only that the design is sufficient to guarantee a predetermined H_∞-norm bound, and not that any achievable bound can be obtained using such a design. For this reason, not all the conditions appearing in Theorem 1.2 are needed.

The problem here is to derive control laws to stabilize the plant (1.3) and provide an H_∞-norm bound for the closed-loop transfer function matrix from the disturbance $w_e = \binom{w_0}{w}$ to z. By Lemma 2.1, a sufficient condition for a state-feedback control $u = Kx$ to stabilize the plant and guarantee the H_∞-norm bound $\|T\|_\infty \leq \alpha$ is that the feedback gain matrix K satisfy

$$(A + BK)^T X + X(A + BK) + \frac{1}{\alpha^2} X G G^T X + (H^T \; K^T)\begin{pmatrix} H \\ K \end{pmatrix} = 0 \tag{2.7}$$

with $X \geq 0$. Rearrange (2.7) as

$$A^T X + XA + \frac{1}{\alpha^2} XGG^T X - XBB^T X + (K^T + XB)(K + B^T X) + H^T H = 0. \qquad (2.8)$$

Setting $K = -B^T X$ in (2.8) gives the state-feedback design equation (1.5). By Lemma 2.1, if $X \geq 0$ solves (2.7), the control law (1.6) results in a closed-loop system with an H_∞-norm bound of α. (Note that the detectability condition of Lemma 2.1 is satisfied if (A, H) is a detectable pair.)

In the output-feedback case, an observer-based control law will be used to approximate a state-feedback control $u = Kx$. To mimic the dynamics of the plant (1.3), the observer takes the form

$$\dot{\xi} = A\xi + Bu + G\hat{w}_0 + L(y - C\xi), \quad u = K\xi, \qquad (2.9a)$$

where a state-feedback model of the disturbance w_0 is assumed as

$$\hat{w}_0 = K_d \xi. \qquad (2.9b)$$

The feedback gain K, observer gain L, and disturbance-estimate gain K_d will be chosen so that, when controller (2.9) is applied to the plant (1.3), the closed-loop system will satisfy the hypotheses of Lemma 2.1.

Introduce the error vector $e = \xi - x$, and write the closed-loop system as

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A + BK & BK \\ GK_d & A + GK_d - LC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} G & 0 \\ -G & L \end{pmatrix} \begin{pmatrix} w_0 \\ w \end{pmatrix} \equiv \tilde{F}_e x_e + \tilde{G}_e w_e, \qquad (2.10a)$$

$$z = \begin{pmatrix} H & 0 \\ K & K \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} \equiv \tilde{H}_e x_e. \qquad (2.10b)$$

Similar to the state-feedback case, the goal is to find $\tilde{X}_e \geq 0$ such that

$$\tilde{F}_e^T \tilde{X}_e + \tilde{X}_e \tilde{F}_e + \frac{1}{\alpha^2} \tilde{X}_e \tilde{G}_e \tilde{G}_e^T \tilde{X}_e + \tilde{H}_e^T \tilde{H}_e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \qquad (2.11)$$

To ensure decoupling of (2.11) into a state-feedback design ARE and an observer design equation, look for a block-diagonal solution

$$\tilde{X}_e = \begin{pmatrix} X & 0 \\ 0 & X_1 \end{pmatrix} \geq 0.$$

Then, the upper-left block of (2.11) is exactly Equation (2.7). If, as in the state-feedback solution, $X \geq 0$ solves (1.5) and the feedback gain is given by

$$K = -B^T X, \tag{2.12}$$

then the upper-left block of (2.11) is satisfied. The upper-right block of (2.11) then gives

$$-XBB^T X + K_d^T G^T X_1 - \frac{1}{\alpha^2} XGG^T X_1 + XBB^T X = 0,$$

which is satisfied if

$$K_d = \frac{1}{\alpha^2} G^T X. \tag{2.13}$$

Given the choices (2.12) and (2.13), the lower-right block of (2.11) gives

$$\begin{aligned} X_1(A + \alpha^{-2}GG^T X - LC) \; &+ \; (A + \alpha^{-2}GG^T X - LC)^T X_1 \\ &+ \; \tfrac{1}{\alpha^2} X_1(GG^T + LL^T)X_1 + XBB^T X = 0. \end{aligned} \tag{2.14}$$

Add to (2.14) the design equation (1.5) to obtain the ARE

$$\begin{aligned} (X + X_1)A \; &+ \; A^T(X + X_1) + \tfrac{1}{\alpha^2}(X + X_1)GG^T(X + X_1) - \alpha^2 C^T C + H^T H \\ &+ \; \left(\tfrac{1}{\alpha} X_1 L - \alpha C^T\right)\left(\tfrac{1}{\alpha} L^T X_1 - \alpha C\right) = 0, \end{aligned} \tag{2.15}$$

which suggests the choice for the observer gain L as

$$X_1 L = \alpha^2 C^T. \tag{2.16}$$

In order that L satisfying (2.16) is guaranteed to exist, impose the restriction $X_1 > 0$. Now introduce

$$Y = \alpha^2 (X + X_1)^{-1} > 0 \tag{2.17}$$

to transform (2.15) into the design ARE (1.7). A solution $Y > 0$ of (1.7), with $\alpha^2 Y^{-1} \geq X$, guarantees $X_e \geq 0$ solves (2.11) when gains K, K_d, and L are computed from (2.12), (2.13), and (2.16). Hence, by Lemma 2.1 the closed-loop transfer-function matrix $T(s) = \tilde{H}_e(sI - \tilde{F}_e)^{-1}\tilde{G}_e$ satisfies $\|T\|_\infty \leq \alpha$, provided $(\tilde{F}_e, \tilde{H}_e)$ is a detectable pair.

The needed detectability condition is satisfied if (A, H) is a detectable pair and $A_\alpha = A + \alpha^{-2}GG^T X - BB^T X$ is Hurwitz. To see this, let $v^T = (v_1^T \;\; v_2^T)$ satisfy

$$\tilde{F}_e v = \begin{pmatrix} A - BB^T X & -BB^T X \\ \alpha^{-2}GG^T X & A + \alpha^{-2}GG^T X - LC \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda v, \tag{2.18}$$

$$\tilde{H}_e v = \begin{pmatrix} H & 0 \\ -B^T X & -B^T X \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0, \tag{2.19}$$

and try to show that $Re(\lambda) < 0$. The upper half of (2.18) and the lower half of (2.19) give $Av_1 = \lambda v_1$, while the upper part of (2.19) gives $Hv_1 = 0$. Since (A, H) is assumed a detectable pair, this implies either $Re(\lambda) < 0$ or $v_1 = 0$. Suppose $v_1 = 0$; then the lower half of (2.18) gives

$$\left(A + \frac{1}{\alpha^2} GG^T X - LC\right) v_2 = \lambda v_2. \tag{2.20}$$

Therefore, pre-multiplying (2.14) by v_2^* and post-multiplying by v_2, and using (2.16), gives

$$2Re(\lambda) v_2^* X_1 v_2 + \frac{1}{\alpha^2} v_2^* X_1 GG^T X_1 v_2 + \alpha^2 v_2^* C^T C v_2 + v_2^* X BB^T X v_2 = 0. \tag{2.21}$$

Since every term but the first in (2.21) is nonnegative, the first term gives

$$Re(\lambda) v_2^* X_1 v_2 \leq 0. \tag{2.22}$$

If inequality holds in (2.22), then $Re(\lambda) < 0$. If equality holds in (2.22), then every term in (2.21) is zero. Hence, $Cv_2 = 0$ and $B^T X v_2 = 0$, and thus (2.20) gives

$$\left(A + \alpha^{-2} GG^T X - BB^T X\right) v_2 = \lambda v_2.$$

By assumption, $A + \alpha^{-2} GG^T X - BB^T X$ is Hurwitz; therefore, $Re(\lambda) < 0$.

The following theorem summarizes the result.

Theorem 2.1. *Suppose (A, H) is a detectable pair, $X \geq 0$ satisfies the state-feedback design ARE (1.5) with $A_\alpha = A + \alpha^{-2} GG^T X - BB^T X$ Hurwitz, and $Y > 0$ satisfies the observer design ARE (1.7) with $\sigma_{\max}\{YX\} < \alpha^2$. If the observer gain is given by*

$$L = (I - \alpha^{-2} YX)^{-1} YC^T, \tag{2.23}$$

then the dynamic controller

$$\dot{\xi} = \left(A + \frac{1}{\alpha^2} GG^T X - BB^T X - LC\right) \xi + Ly \tag{2.24}$$

$$u = -B^T X \xi \tag{2.25}$$

stabilizes the plant (1.3), and the closed-loop transfer-function matrix $T(s) = \tilde{H}_e (sI - \tilde{F}_e)^{-1} \tilde{G}_e$ satisfies $\|T\|_\infty \leq \alpha$.

2.4 The Approach to Computing Families of Controllers

Lemma 2.1 provides not only a new way of designing control laws which guarantee stability and an H_∞-norm bound, but also a method of characterizing families of such controllers. Such a characterization is based on the following convexity property of the matrix Riccati function R defined in (1.17): If there are several matrices $X_i \geq 0$ which satisfy the ARE $R(X_i) = 0$, then any convex combination Z of the X_i's satisfies $R(Z) \leq 0$. This fact allows easy computation of a family of matrices $Z \geq 0$ satisfying

$$A^T Z + ZA + \frac{1}{\alpha^2} ZGG^T Z - ZBB^T Z + H^T H \leq 0.$$

For any such Z, it is shown that the state-feedback control $u = -B^T Zx$ provides stability and the H_∞-norm bound $\|T\|_\infty \leq \alpha$.

Any member of the family of state-feedback controls can be used as a reference for centralized or decentralized observer-based controls. In the centralized case, the convexity property of the dual-form Riccati function associated with observer design also permits computation of a family of observer gain matrices. In the decentralized case, however, the Riccati-like design equation does not allow this freedom. As a result, the freedom in computing the family of decentralized control laws is just that freedom available in computing the family of state-feedback control laws.

The remainder of the thesis develops the various new control design methods based on Lemma 2.1, starting with the basic decentralized design, and proceeding to robust and reliable designs, and finally to families of designs, all guaranteeing predetermined levels of worst-case (H_∞) disturbance attenuation.

Chapter 3

The Basic Decentralized Control Design

The same approach applied to the centralized control problem in Section 2.3 is now applied to the decentralized problem. The design derived here also appears in [30].

3.1 Design Derivation

Consider the plant (1.14) with (A, H) a detectable pair. For convenience, adopt the following notation:

$$\sum_{i=1}^{q} B_i u_i = (B_1\ B_2 \ldots B_q)\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{pmatrix} \equiv Bu, \tag{3.1a}$$

$$y \equiv \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{pmatrix} x + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_q \end{pmatrix} \equiv Cx + w, \tag{3.1b}$$

$$w_e \equiv \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_q \end{pmatrix} = \begin{pmatrix} w_0 \\ w \end{pmatrix}, \tag{3.1c}$$

$$S_i = B_i B_i^T, \quad i \in \{1, 2, \ldots, q\}, \tag{3.1d}$$

$$S = S_1 + S_2 + \ldots + S_q = BB^T. \tag{3.1e}$$

The problem is to design a controller for each of the q control channels, where the i^{th} controller uses the local measurement y_i to generate the local control u_i for the plant.

The basic decentralized control law to be developed stabilizes the plant and provides a predetermined H_∞-norm bound for the closed-loop transfer-function matrix from w_e to z. The controllers which make up the control law are based on observers which form estimates ξ_i, $i \in \{1, 2, \ldots, q\}$, of the state x for feedback. The state estimates are used for feedback so as to approximate the state-feedback control

$$u = -B^T X x, \tag{3.2}$$

where $X \geq 0$ satisfies the ARE

$$A^T X + XA + \frac{1}{\alpha^2} X G G^T X - XSX + H^T H = 0. \tag{3.3}$$

That is, the i^{th} control is given by

$$u_i = -B_i^T X \xi_i, \tag{3.4}$$

which approximates a subvector of the state-feedback control (3.2). To mimic the plant dynamics (1.14a), the i^{th} observer should ideally have the form

$$\dot{\xi}_i = A\xi_i + \sum_{j=1}^{q} B_j u_j + G w_0 + L_i (y_i - C_i \xi_i), \tag{3.5}$$

where L_i is some observer gain matrix. However, since the disturbance w_0 and the controls u_j, $j \neq i$, are not available to the observer, (3.5) cannot be implemented directly. Just as the centralized observer (1.11) uses (1.12) as an estimate of the worst disturbance, the i^{th} decentralized observer replaces w_0 in (3.5) by

$$\hat{w}_0^i = \frac{1}{\alpha^2} G^T X \xi_i. \tag{3.6}$$

The i^{th} observer also replaces u_j, $j \neq i$, by

$$\hat{u}_j^i = -B_j^T X \xi_i, \tag{3.7}$$

which are approximations, based on the state estimate of the i^{th} controller, of the controls applied by the other controllers according to their shared strategy. With the control (3.4), the observer structure (3.5), and estimates (3.6) and (3.7), the i^{th} controller becomes

$$\dot{\xi}_i = \left(A + \frac{1}{\alpha^2} G G^T X - SX - L_i C_i\right) \xi_i + L_i y_i \tag{3.8a}$$

$$u_i = -B_i^T X \xi_i, \tag{3.8b}$$

where the observer gains L_i, $i \in \{1, 2, \ldots, q\}$, are to be determined.

Applying the q controllers (3.8) to the plant (1.14) gives a closed-loop system of order $(q+1)n$ described by

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & -BB_c^T X_c \\ L_c C & A_{\alpha c} - L_c C_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & L_c \end{pmatrix} \begin{pmatrix} w_0 \\ w \end{pmatrix} \equiv F_e x_e + G_e w_e \tag{3.9a}$$

$$z = \begin{pmatrix} H & 0 \\ 0 & -B_c^T X_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \equiv H_e x_e, \tag{3.9b}$$

where $\xi^T = (\xi_1^T \; \xi_2^T \; \ldots \; \xi_q^T)$, and

$$A_{\alpha c} = \text{Diag}\,(A_\alpha, A_\alpha, \ldots, A_\alpha) \tag{3.10a}$$

$$A_\alpha = A + \frac{1}{\alpha^2} G G^T X - SX, \tag{3.10b}$$

$$B_c = \text{Diag}\,(B_1, B_2, \ldots, B_q), \tag{3.10c}$$

$$C_c = \text{Diag}\,(C_1, C_2, \ldots, C_q), \tag{3.10d}$$

$$L_c = \text{Diag}\,(L_1, L_2, \ldots, L_q), \tag{3.10e}$$

$$X_c = \text{Diag}\,(X, X, \ldots, X). \tag{3.10f}$$

For convenience, define also

$$I_c^T = [I \; I \; \ldots \; I] \in \mathbb{R}^{n \times qn}, \tag{3.10g}$$

$$G_c = I_c G, \tag{3.10h}$$

$$A_c = A_{\alpha c} + I_c B B_c^T X_c. \tag{3.10i}$$

Then, transforming coordinates of (3.9) such that the last qn state variables are the errors $e_i = \xi_i - x, \quad i \in \{1, 2, \ldots, q\}$, gives

$$\dot{\tilde{x}}_e = \tilde{F}_e \tilde{x}_e + \tilde{G}_e w_e, \quad z = \tilde{H}_e \tilde{x}_e,$$

where

$$\tilde{F}_e = M_e^{-1} F_e M_e = \begin{pmatrix} A - SX & -BB_c^T X_c \\ \alpha^{-2} G_c G^T X & A_c - L_c C_c \end{pmatrix}, \; \tilde{G}_e = M_e^{-1} G_e = \begin{pmatrix} G & 0 \\ -G_c & L_c \end{pmatrix}, \tag{3.11a}$$

$$\tilde{H}_e = H_e M_e = \begin{pmatrix} H & 0 \\ -B^T X & -B_c^T X_c \end{pmatrix}, \; M_e = \begin{pmatrix} I & 0 \\ I_c & I \end{pmatrix}. \tag{3.11b}$$

The existence of a $(q+1)n \times (q+1)n$ matrix $X_e \geq 0$ satisfying

$$\tilde{F}_e^T \tilde{X}_e + \tilde{X}_e \tilde{F}_e + \frac{1}{\alpha^2} \tilde{X}_e \tilde{G}_e \tilde{G}_e^T \tilde{X}_e + \tilde{H}_e^T \tilde{H}_e = 0 \tag{3.12}$$

will by Lemma 2.1 guarantee stability and an H_∞-norm bound for the closed-loop system (3.9). Assume the form

$$\tilde{X}_e = \begin{pmatrix} X & 0 \\ 0 & X_1 \end{pmatrix}, \tag{3.13}$$

with $X \geq 0$ solving (3.3) and $X_1 > 0$ undetermined, and decompose the left-hand side of (3.12) into appropriately sized blocks as

$$\tilde{F}_e^T \tilde{X}_e + \tilde{X}_e \tilde{F}_e + \frac{1}{\alpha^2} \tilde{X}_e \tilde{G}_e \tilde{G}_e^T \tilde{X}_e + \tilde{H}_e^T \tilde{H}_e = \begin{pmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{pmatrix}. \tag{3.14}$$

Then, it turns out that the off-diagonal block U_{12} is identically zero, and that (3.3) gives $U_{11} = 0$. Hence, independent of L_c and X_1, (3.14) becomes

$$\tilde{F}_e^T \tilde{X}_e + \tilde{X}_e \tilde{F}_e + \frac{1}{\alpha^2} \tilde{X}_e \tilde{G}_e \tilde{G}_e^T \tilde{X}_e + \tilde{H}_e^T \tilde{H}_e = \begin{pmatrix} 0 & 0 \\ 0 & U_{22} \end{pmatrix},$$

with

$$U_{22} = (A_c - L_c C_c)^T X_1 + X_1 (A_c - L_c C_c) + \frac{1}{\alpha^2} X_1 (G_c G_c^T + L_c L_c^T) X_1 + X_c B_c B_c^T X_c.$$

Defining $W = \alpha^2 X_1^{-1}$, this reduces to

$$\begin{aligned} U_{22} &= \tfrac{1}{\alpha^2} X_1 \{ W A_c^T + A_c W + \tfrac{1}{\alpha^2} W X_c B_c B_c^T X_c W - W C_c^T C_c W \\ &+ G_c G_c^T + (L_c - W C_c^T)(L_c^T - C_c W) \} X_1. \end{aligned} \tag{3.15}$$

It is now possible to pick X_1 (or, equivalently, W) and L_c such that $U_{22} = 0$. While it is logical in view of Lemma 2.1 to try to eliminate the last term in (3.15), this is not generally possible, since L_c must be block-diagonal. Thus, L_c is chosen to eliminate the $n \times n$ main-diagonal blocks of $L_c - W C_c^T$. This requires

$$L_c = W_D C_c^T, \tag{3.16}$$

where W_D is given by

$$W = \begin{pmatrix} W_{11} & W_{12} & \dots & W_{1q} \\ W_{21} & W_{22} & \dots & W_{2q} \\ \vdots & \vdots & & \vdots \\ W_{q1} & W_{q2} & \dots & W_{qq} \end{pmatrix}, \; W_D = \text{Diag}\,(W_{11}, W_{22}, \dots, W_{qq}),$$

or

$$L_i = W_{ii} C_i^T, \quad i \in \{1, 2, \dots, q\}. \tag{3.17}$$

Then, (3.15) becomes

$$\begin{aligned} U_{22} &= \tfrac{1}{\alpha^2} X_1 \{ W A_c^T + A_c W + \tfrac{1}{\alpha^2} W X_c B_c B_c^T X_c W - W C_c^T C_c W + G_c G_c^T \\ &+ (W - W_D) C_c^T C_c (W - W_D) \} X_1. \end{aligned} \tag{3.18}$$

Therefore, if $W > 0$ satisfies the Riccati-like algebraic equation

$$\begin{gathered} W A_c^T + A_c W + \tfrac{1}{\alpha^2} W X_c B_c B_c^T X_c W - W C_c^T C_c W \\ + G_c G_c^T + (W - W_D) C_c^T C_c (W - W_D) = 0, \end{gathered} \tag{3.19}$$

then $U_{22} = 0$, and (3.12) is satisfied. Since $W > 0$ is required, $\tilde{X}_e \geq 0$ holds automatically; therefore, provided $(\tilde{F}_e, \tilde{H}_e)$ is a detectable pair, Lemma 2.1 gurantees that $\tilde{F}_e$ is Hurwitz and that $T(s) = \tilde{H}_e(sI - \tilde{F}_e)^{-1} \tilde{G}_e$ satisfies $\|T\|_\infty \leq \alpha$. The following lemma establishes the detectability condition.

Lemma 3.1. *Given the definitions (3.10) and (3.11), where $X \geq 0$ satisfies (3.3), $W > 0$ satisfies (3.19), and L_c satisfies (3.16), the pair $(\tilde{F}_e, \tilde{H}_e)$ is detectable under the following three conditions:*

(i) (A, H) *is a detectable pair;*

(ii) $A_\alpha \equiv A + \alpha^{-2}GG^TX - SX$ *is Hurwitz;*

(iii) $A_\alpha + SX$ *has no eigenvalues on the* $j\omega$*-axis.*

Proof: Suppose λ is an eigenvalue of $\tilde{F}_e$ corresponding to an unobservable mode of $(\tilde{F}_e, \tilde{H}_e)$; that is, some $v^T = (v_1^T \; v_2^T) \neq 0$ satisfies

$$\tilde{F}_e v = \begin{pmatrix} A - BB^TX & -BB_c^TX_c \\ \alpha^{-2}G_cG^TX & A_c - L_cC_c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda v \tag{3.20}$$

and

$$\tilde{H}_e v = \begin{pmatrix} H & 0 \\ -B^TX & -B_c^TX_c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0. \tag{3.21}$$

The proof now consists of showing that $Re(\lambda) < 0$.

The lower block of (3.21) and the upper block of (3.20) combine to give $Av_1 = \lambda v_1$, while the upper block of (3.21) gives $Hv_1 = 0$. Since (A, H) is assumed a detectable pair, this implies that either $Re(\lambda) < 0$ or $v_1 = 0$. If $v_1 = 0$, then the lower block of (3.20) gives

$$(A_c - L_cC_c)v_2 = \lambda v_2. \tag{3.22}$$

The detectability proof is completed by showing that $A_c - L_cC_c$ is Hurwitz. The bracketed expression in (3.15) is equal to zero; therefore

$$(A_c - L_cC_c)W + W(A_c - L_cC_c)^T + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW + G_cG_c^T + L_cL_c^T = 0. \tag{3.23}$$

Let η^* be a left-eigenvector of $A_c - L_cC_c$ corresponding to the eigenvalue λ. Multiply (3.23) on the left by η^* and on the right by η to obtain

$$2Re(\lambda)\eta^*W\eta + \frac{1}{\alpha^2}\eta^*WX_cB_cB_c^TX_cW\eta + \eta^*G_cG_c^T\eta + \eta^*L_cL_c^T\eta = 0. \tag{3.24}$$

Since every other term in (3.24) is nonnegative, $Re(\lambda)\eta^*W\eta \leq 0$, with $W > 0$ assumed; therefore, $Re(\lambda) \leq 0$. The following argument demonstrates that $Re(\lambda) \neq 0$. If $Re(\lambda) = 0$, then every term in (3.24) must be zero; hence, $\eta^*L_c = 0$. Then λ is an eigenvalue of A_c. But a similarity

transformation on A_c reveals that it can have no imaginary eigenvalues: If

$$M = \begin{pmatrix} I & & & \\ -I & I & & \\ \vdots & \vdots & \ddots & \\ -I & 0 & \dots & I \end{pmatrix},$$

then

$$M^{-1}A_cM = \begin{pmatrix} A_\alpha + SX & S_2X & \dots & S_qX \\ & A_\alpha & \dots & 0 \\ & & \ddots & \vdots \\ & & & A_\alpha \end{pmatrix},$$

where A_α is assumed Hurwitz, and $A_\alpha + SX$ is assumed to have no imaginary eigenvalues. Q.E.D.

Under the conditions of Lemma 3.1, $\tilde{F}_e$ is Hurwitz by Lemma 2.1. Therefore, F_e is also Hurwitz, and the closed-loop transfer-function matrix $T(s) = H_e(sI - F_e)^{-1}G_e = \tilde{H}_e(sI - \tilde{F}_e)^{-1}\tilde{G}_e$ from w_e to z satisfies $\|T\|_\infty \le \alpha$. Condition (iii) of Lemma 3.1 is a new technical condition which must be introduced for the decentralized control problem.

The following theorem summarizes the result:

Theorem 3.1. *Let (A, H) be a detectable pair and α be a positive scalar. Suppose $X \geq 0$ satisfies*

$$A^TX + XA + \frac{1}{\alpha^2}XGG^TX - XSX + H^TH = 0, \tag{3.3}$$

$A_\alpha \equiv A + \alpha^{-2}GG^TX - SX$ is Hurwitz, and $A_\alpha + SX$ has no $j\omega$-axis eigenvalues. Let $W > 0$ satisfy the Riccati-like algebraic equation

$$\begin{aligned} WA_c^T \;&+\; A_cW + \tfrac{1}{\alpha^2}WX_cB_cB_c^TX_cW - WC_c^TC_cW \\ &+\; G_cG_c^T + (W - W_D)C_c^TC_c(W - W_D) = 0. \end{aligned} \tag{3.19}$$

If the observer gains L_i, $i \in \{1, 2, \dots, q\}$, are given by

$$L_i = W_{ii}C_i^T, \tag{3.17}$$

then the decentralized feedback control law

$$\dot{\xi}_i = \left(A + \frac{1}{\alpha^2}GG^TX - SX - L_iC_i\right)\xi_i + L_iy_i, \quad i \in \{1, 2, \dots, q\}, \tag{3.8a}$$

$$u_i = -B_i^T X \xi_i, \quad i \in \{1, 2, \ldots, q\}, \tag{3.8b}$$

stabilizes the plant (1.14), and the closed-loop transfer-function matrix

$$T(s) = H_e(sI - F_e)^{-1} G_e$$

from w_e to z (with F_e, G_e, and H_e defined in (3.9)) satisfies

$$\|T\|_\infty \leq \alpha.$$

3.2 Example 1

Consider the plant (1.14) with $q = 2$ and

$$A = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 2 \\ -1 & 0 & -2 & -3 \\ -2 & -1 & 2 & -1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad G = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$C_1 = [1\ 0\ 0\ 0] \quad C_2 = [0\ 0\ 1\ 0] \quad H = [1\ 0\ -1\ 0].$$

The spectrum of A is $\{-2.56, -1.32 \pm j2.92, +0.19\}$; hence, the plant has an unstable mode.

To compute a decentralized control for this plant, first form the coefficients of (3.19) from the plant matrices and the state-feedback design equation solution. Then, solve (3.19) by an iterative method: Compute an approximate solution W_0 by ignoring the complicating term $Q = (W - W_D)C_c^T C_c(W - W_D)$. Then use W_0 to compute an approximation of Q_0 of Q, and use Q_0 in the obvious way to compute the next approximate solution W_1. Iterate this procedure until the candidate solution W_i makes the matrix norm of left-hand side of (3.19) less than some acceptable tolerance; then take W_i as the solution W of (3.19). The tolerance used for this example was 0.001.

Table 3.1 compares the closed-loop eigenvalues and H_∞ norms of state-feedback designs with those of decentralized observer-based control designs for several values of α. For $\alpha > 4$, the state-feedback eigenvalues are easily recognizable in the spectra of the decentralized-control systems; for smaller α, more interaction with other poles is evident. The sequence of candidate solutions of the Riccati-like equation converges for $\alpha \geq 2$, while the state-feedback design Riccati equation has an appropriate solution for $\alpha \geq 1.3$.

Table 3.1. Closed-loop spectra and H_∞ norms for varying α.

	State Feedback		Decentralized Control	
	Spectrum	$\|T\|_\infty$	Spectrum	$\|T\|_\infty$
$\alpha = 20$	-0.24 -2.54 -1.45$\pm j$2.98	2.30	-0.24 -2.52 -1.26$\pm j$2.90 -0.38 -2.54 -1.47$\pm j$2.97 -1.07 -2.70 -1.45$\pm j$2.98	3.64
$\alpha = 16$	-0.24 -2.54 -1.45$\pm j$2.98	2.30	-0.24 -2.52 -1.26$\pm j$2.90 -0.38 -2.54 -1.47$\pm j$2.97 -1.08 -2.70 -1.45$\pm j$2.98	3.63
$\alpha = 12$	-0.24 -2.54 -1.45$\pm j$2.98	2.29	-0.25 -2.52 -1.26$\pm j$2.90 -0.38 -2.54 -1.47$\pm j$2.97 -1.08 -2.70 -1.45$\pm j$2.98	3.59
$\alpha = 8$	-0.24 -2.54 -1.45$\pm j$2.98	2.27	-0.27 -2.52 -1.26$\pm j$2.90 -0.37 -2.54 -1.47$\pm j$2.97 -1.09 -2.70 -1.45$\pm j$2.98	3.49
$\alpha = 4$	-0.27 -2.54 -1.46$\pm j$2.98	2.15	-0.35$\pm j$0.08 -1.26$\pm j$2.91 -1.18 -2.54 -1.47$\pm j$2.97 -2.49 -2.71 -1.45$\pm j$2.98	3.05
$\alpha = 2$	-0.46 -2.54 -1.46$\pm j$2.98	1.76	-2.36$\pm j$0.85 -1.21$\pm j$2.98 -0.48 -2.53 -1.47$\pm j$2.98 -1.38 -2.79 -1.45$\pm j$2.94	1.995
$\alpha = 1.3$	-2.59 -3.11 -1.45$\pm j$2.94	1.30	none	none

3.3 Example 2

Consider the 5^{th}-order plant (1.14) with $q = 2$ and

$$A = \begin{pmatrix} 0 & 1 & 4 & -4 & 1 \\ -3 & -1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 2 & 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & -2 & -2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The spectrum of A is

$$\Lambda(A) = \{-0.0108 \pm j3.717, -3.7138, -1.5906, +1.3262\};$$

hence, the plant has an unstable mode and a lightly-damped stable mode. This section gives the results of H_∞-norm-bounding control designs for this plant. First, state-feedback solutions are presented, then observer-based solutions, both centralized and decentralized. For various values of the design parameter α, the spectrum, feedback and observer gains, and H_∞ norm for the closed-loop system are given.

3.3.1 State feedback

State-feedback designs can be computed for values of α varying from ∞ to 1.069199. For $\alpha =$ 1.069198, the solution X of the state-feedback design ARE (3.3) has a negative eigenvalue; hence, for all practical purposes, $\alpha_\infty = 1.069199$.

The closed-loop poles are the eigenvalues of $F = A - SX$. Figure 3.1 shows the position of the closed-loop poles for a varying from ∞ to α_∞. Note that as α decreases from ∞ to 2.0, the poles barely move. As α decreases from 2.0 to 1.1, the most oscillatory mode is damped somewhat, and the other complex pole-pair meets at the real axis and splits into a real pair. Finally, as α decreases in the short interval from 1.1 to α_∞, the closed-loop poles are extremely sensitive to variations in α: The two remaining complex poles move leftward in the complex plane and meet at the real axis; then one pole goes toward $-\infty$. Naturally, moving a pole far into the left-half plane requires high feedback gains: The LQ feedback matrix is

$$K_{LQ} = \begin{pmatrix} -0.51 & -1.00 & -0.21 & -0.98 & -0.80 \\ -0.48 & -0.40 & 0.44 & -0.94 & -0.47 \end{pmatrix}$$

with resulting closed-loop spectrum

$$\Lambda(F) = \{-0.92 \pm j3.98, -1.78 \pm j0.35, -3.54\}$$

while a nearly H_∞-optimal ($\alpha = 1.07$) feedback matrix is

$$K = \begin{pmatrix} 13.21 & -3.67 & -51.59 & 61.76 & 7.93 \\ -66.51 & 3.97 & 189.88 & -235.86 & -40.62 \end{pmatrix}$$

with resulting spectrum

$$\Lambda(F) = \{-81.54, -10.51, -3.65, -2.63, -1.57\}.$$

These gains are much larger than the LQ gains, and they also have different signs. Reducing α to $\alpha_\infty = 1.069199$ results in gains (and one closed-loop pole) of magnitude larger than 10^5.

3.3.2 Centralized observer feedback

Observer-based centralized controls can be computed by the method of Theorem 2.1 for values of a ranging from ∞ to 1.913. For $\alpha = 1.912$, the solutions X and Y of (3.3) and (1.7) do not satisfy the condition $\sigma_{\max}\{YX\} < \alpha^2$.

Figure 3.2 shows the position of the closed-loop poles for α varying from ∞ to 1.913. As α falls from ∞ to 3.0, the most oscillatory modes are damped somewhat, and all but the leftmost of the real poles move to the left on the real axis. As α falls from 3.0 to 2.4, the two leftmost real poles meet, split into a complex pair, circle leftward, meet again on the real axis, and move apart. Again, as α approaches its minimum, one pole moves off toward $-\infty$. As α decreases from ∞ to 1.913, each real-axis pole effectively shifts from its original LQG position to the LQG position vacated by the pole to its left, leaving the rightmost LQG position vacant and moving the leftmost real-axis pole toward $-\infty$.

The LQG ($\alpha = \infty$) observer-gain matrix is

$$L_{LQG} = \begin{pmatrix} 1.37 & -0.71 & 0.09 & 0.21 \\ -0.71 & 1.95 & 0.79 & 0.70 \\ 0.40 & 0.24 & 0.26 & 0.31 \\ 0.09 & 0.79 & 1.03 & 0.15 \\ 0.21 & 0.70 & 0.15 & 0.61 \end{pmatrix}$$

with resulting closed-loop spectrum

$$\Lambda(F_e) = \{-0.92 \pm j3.98, -1.78 \pm j0.35, -3.54, -1.09 \pm j3.82, -1.32, -1.69, -3.78\},$$

while the observer-gain matrix for $\alpha = 1.92$ is

$$L = \begin{pmatrix} 2.47 & 1.90 & 3.26 & 1.29 \\ 1.90 & 94.47 & 98.89 & 24.57 \\ 1.55 & 25.95 & 27.67 & 7.12 \\ 3.26 & 98.89 & 105.59 & 25.42 \\ 1.29 & 24.57 & 25.42 & 7.04 \end{pmatrix}$$

with resulting spectrum

$$\Lambda(F_e) = \{-204.31, -1.22 \pm j4.41, -1.47 \pm j3.29, -1.75 \pm j0.42, -3.83, -3.51, -1.65\}.$$

Reducing α to 1.913 results in some gains (and one pole) with magnitudes on the order of 10^3.

3.3.3 Decentralized control

Decentralized controls can be computed using the simple iterative method described in Section 3.2 for values of α ranging from ∞ to 2.3323. The smaller the value of α, the more iterations are required to obtain convergence: For example, to satisfy a tolerance of 0.001 on the largest singular value of the left-hand side of the Riccati-like equation, $\alpha = 10$ requires only 6 iterations, while $\alpha = 2.35$ requires 47 iterations. To speed up computations for small α, the solution for a slightly larger α can be used as the starting point; however, this "imbedding" practice seems to result in convergence of the algorithm only when using the starting point $W = 0$ also results in convergence. For $\alpha = 2.3322$ and below, the algorithm does not seem to converge.

Figure 3.3 shows the position of the closed-loop poles for α varying from ∞ to 2.3323. As α decreases, the oscillatory modes are damped, and the real poles move to the left on the real axis. Again, as α approaches its minimum, the poles on the real axis seem to be shifting left into the positions originally occupied by other poles for $\alpha = \infty$.

For $\alpha = \infty$, the observer-gain matrices are

$$L_1 = \begin{pmatrix} 1.63 & -0.90 \\ -0.90 & 2.61 \\ 0.41 & 0.40 \\ -0.04 & 1.33 \\ 0.32 & 0.65 \end{pmatrix}, \; L_2 = \begin{pmatrix} 0.07 & -0.31 \\ 0.98 & 1.15 \\ 0.31 & 0.44 \\ 1.38 & 0.16 \\ 0.16 & 1.22 \end{pmatrix},$$

while the observer-gain matrices for $\alpha = 2.3323$ are

$$L_1 = \begin{pmatrix} 3.03 & -3.17 \\ -3.17 & 19.26 \\ 0.33 & 3.71 \\ -2.16 & 18.89 \\ 0.76 & 2.92 \end{pmatrix}, \; L_2 = \begin{pmatrix} 5.45 & 0.97 \\ 10.08 & 6.02 \\ 3.58 & 1.86 \\ 11.80 & 2.40 \\ 2.40 & 3.48 \end{pmatrix}.$$

Since the solution for $\alpha = 2.3323$ displays somewhat higher gains and an eigenvalue moving to the left, it seems a reasonable hypothesis that solutions may exist for smaller α, giving a high-gain result as in the state-feedback and centralized observer cases.

3.3.4 Spectrum and H_∞ norm comparisons

The spectra for state-feedback solutions and subspectra for centralized and decentralized observer-based solutions are shown for various values of α in Table 3.2. The state-feedback poles are recognizable among the poles of both observer-based solutions. Although the state-feedback root-locus plot (Fig. 3.1) appears quite different from the other two (Figs. 3.2 and 3.3), the observer-based solutions no longer exist when α is small enough that the state-feedback poles have moved significantly from their LQ positions.

The H_∞ norms of the closed-loop systems are compared for $\alpha \leq 5$ in Figure 3.4. The norms are seen to be monotone increasing with α. For $\alpha = \infty$, the H_∞ norms are $\|T\|_\infty = 1.55$ for state feedback, $\|T\|_\infty = 3.322$ for centralized observer feedback, and $\|T\|_\infty = 4.61$ for decentralized observer feedback, where $T(s)$ is the closed-loop transfer function matrix in each case. As the theory guarantees, the H_∞ norms are always smaller than the design parameter α. In the state-feedback and centralized observer-based designs, the actual H_∞ norms and the bound α are very close for α close to the minimum value. In the decentralized case, the actual norm approaches the bound α in the neighborhood of $\alpha = 2.5$, then falls away slightly from the bound as α approaches the minimum value for which solutions of the Riccati-like design equation were computed. The "slack" in the bound suggests that decentralized designs guaranteeing smaller norms may exist, possibly corresponding to solutions of the Riccati-like equation for smaller values of α. Such solutions would have to be obtained by methods different from those used in this example.

Table 3.2. Closed-loop eigenvalues.

	State Feedback	Centralized Output Feedback	Decentralized Control
$\alpha = \infty$	$-0.92 \pm j3.89$ $-1.78 \pm j0.35$ -3.54	$-0.92 \pm j3.89$ $-1.78 \pm j0.35$ $-3.54 \quad \ldots$	$-0.92 \pm j3.89$ $-1.78 \pm j0.35$ $-3.54 \quad \ldots$
$\alpha = 10$	$-0.92 \pm j3.89$ $-1.78 \pm j0.35$ -3.54	$-0.95 \pm j3.93$ $-1.77 \pm j0.34$ $-3.54 \quad \ldots$	$-0.88 \pm j3.94$ $-1.77 \pm j0.34$ $-3.54 \quad \ldots$
$\alpha = 5$	$-0.94 \pm j3.89$ $-1.78 \pm j0.35$ -3.54	$-0.97 \pm j4.03$ $-1.71 \pm j0.35$ $-3.55 \quad \ldots$	$-0.87 \pm j4.02$ $-1.74 \pm j0.32$ $-3.56 \quad \ldots$
$\alpha = 3$	$-0.99 \pm j3.89$ $-1.78 \pm j0.35$ -3.54	$-1.01 \pm j4.17$ $-1.73 \pm j0.43$ $-3.60 \quad \ldots$	$-0.86 \pm j4.18$ $-1.66 \pm j0.46$ $-3.58 \quad \ldots$
$\alpha = 2.5$	$-1.03 \pm j3.89$ $-1.78 \pm j0.36$ -3.54	$-1.04 \pm j4.26$ $-1.74 \pm j0.43$ $-3.46 \quad \ldots$	$-0.90 \pm j4.35$ $-1.72 \pm j0.46$ $-3.67 \quad \ldots$
$\alpha = 2$	$-1.12 \pm j3.89$ $-1.78 \pm j0.36$ -3.54	$-1.17 \pm j4.39$ $-1.75 \pm j0.42$ $-3.51 \quad \ldots$	

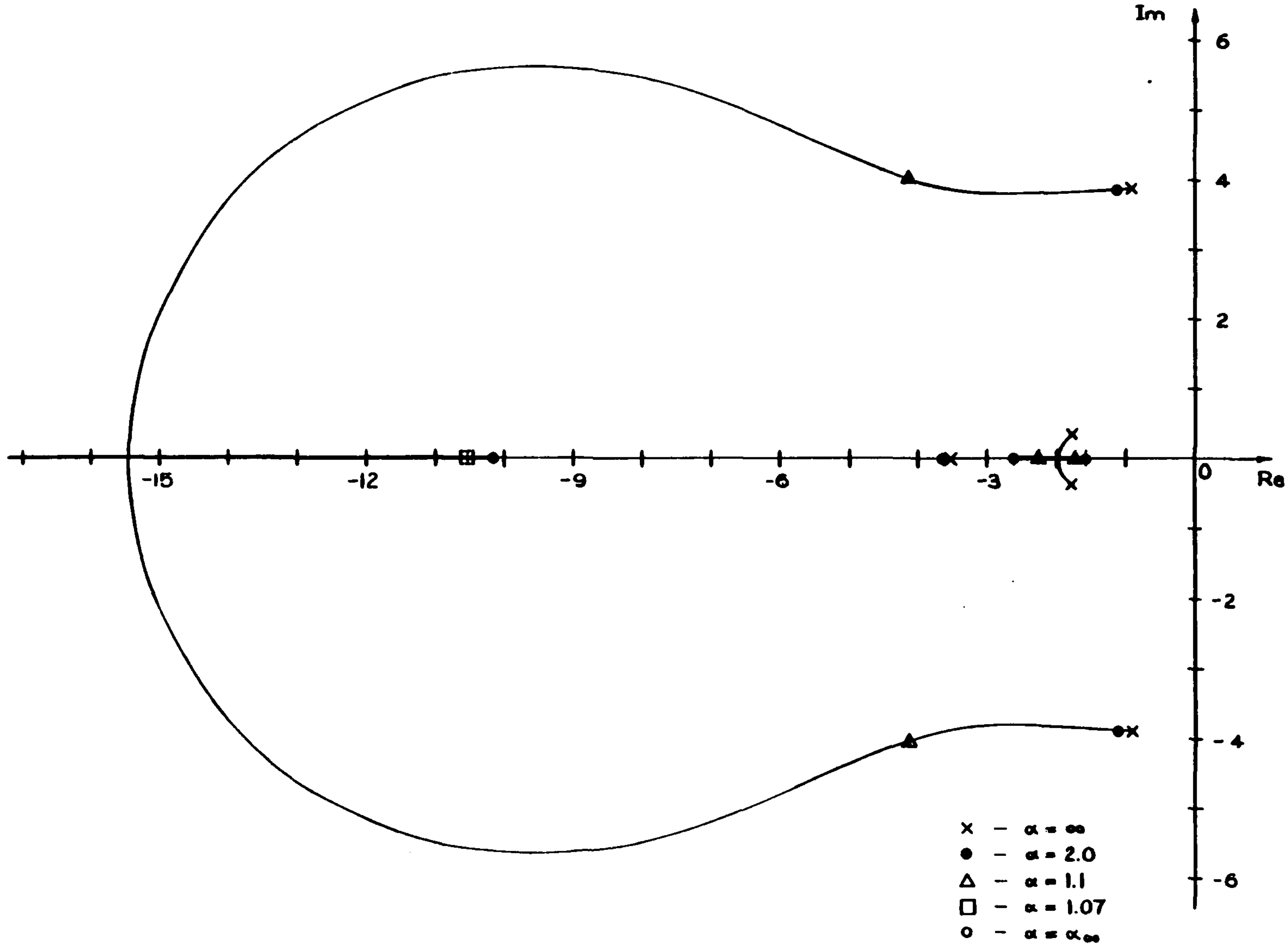


Figure 3.1: State-feedback poles for varying α, Example 2.

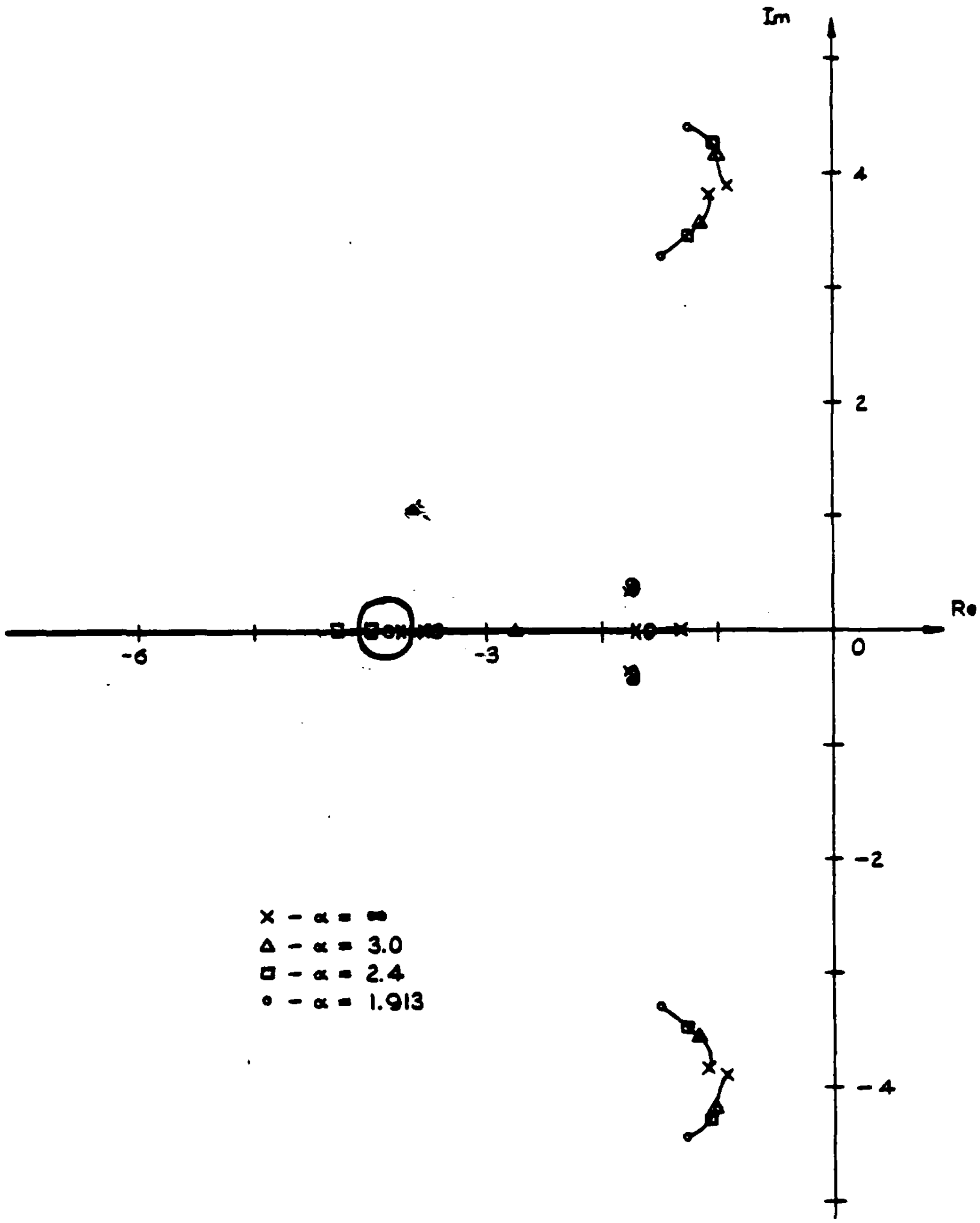


Figure 3.2: Output-feedback poles for varying α, Example 2.

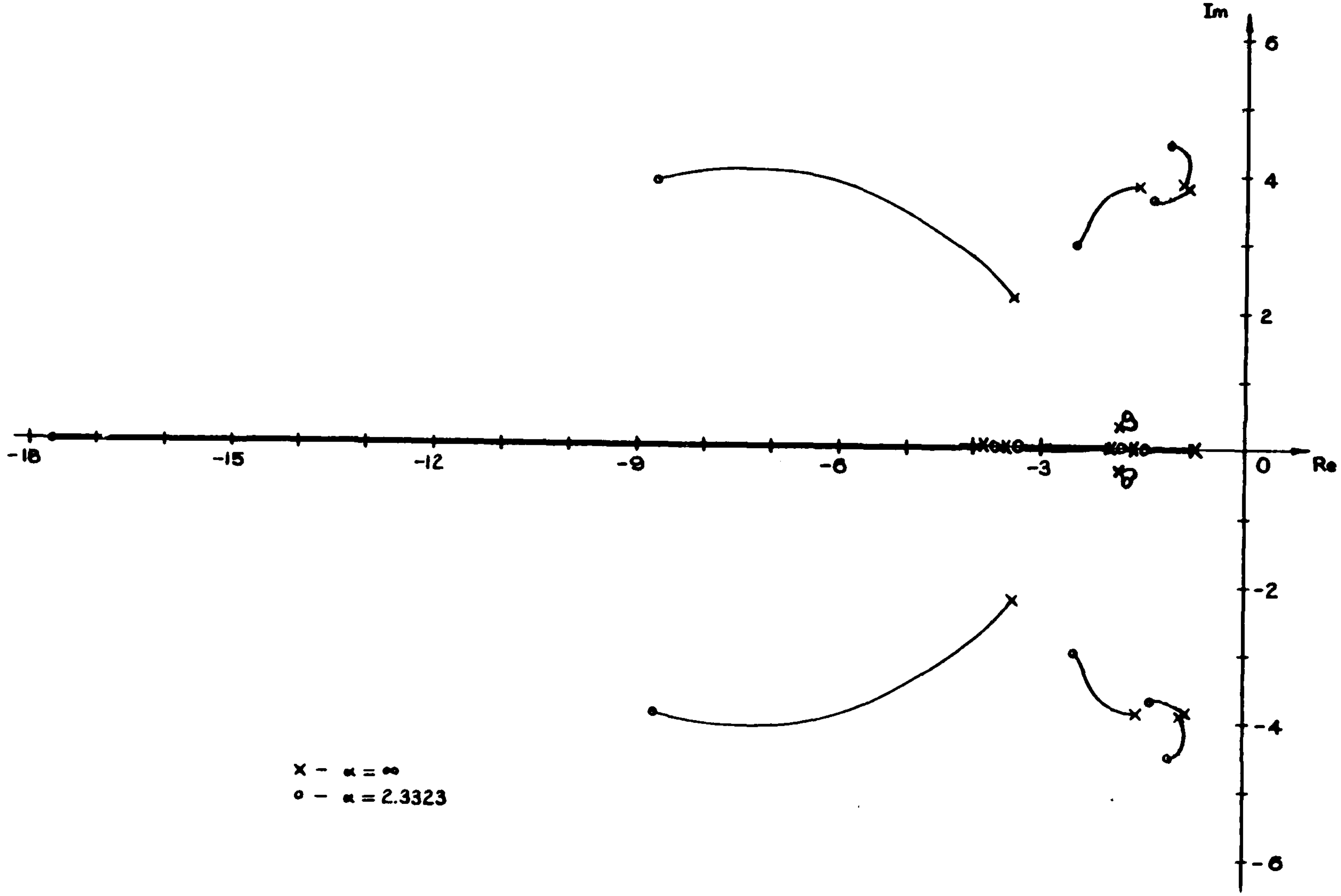


Figure 3.3: Closed-loop poles for decentralized control, Example 2.

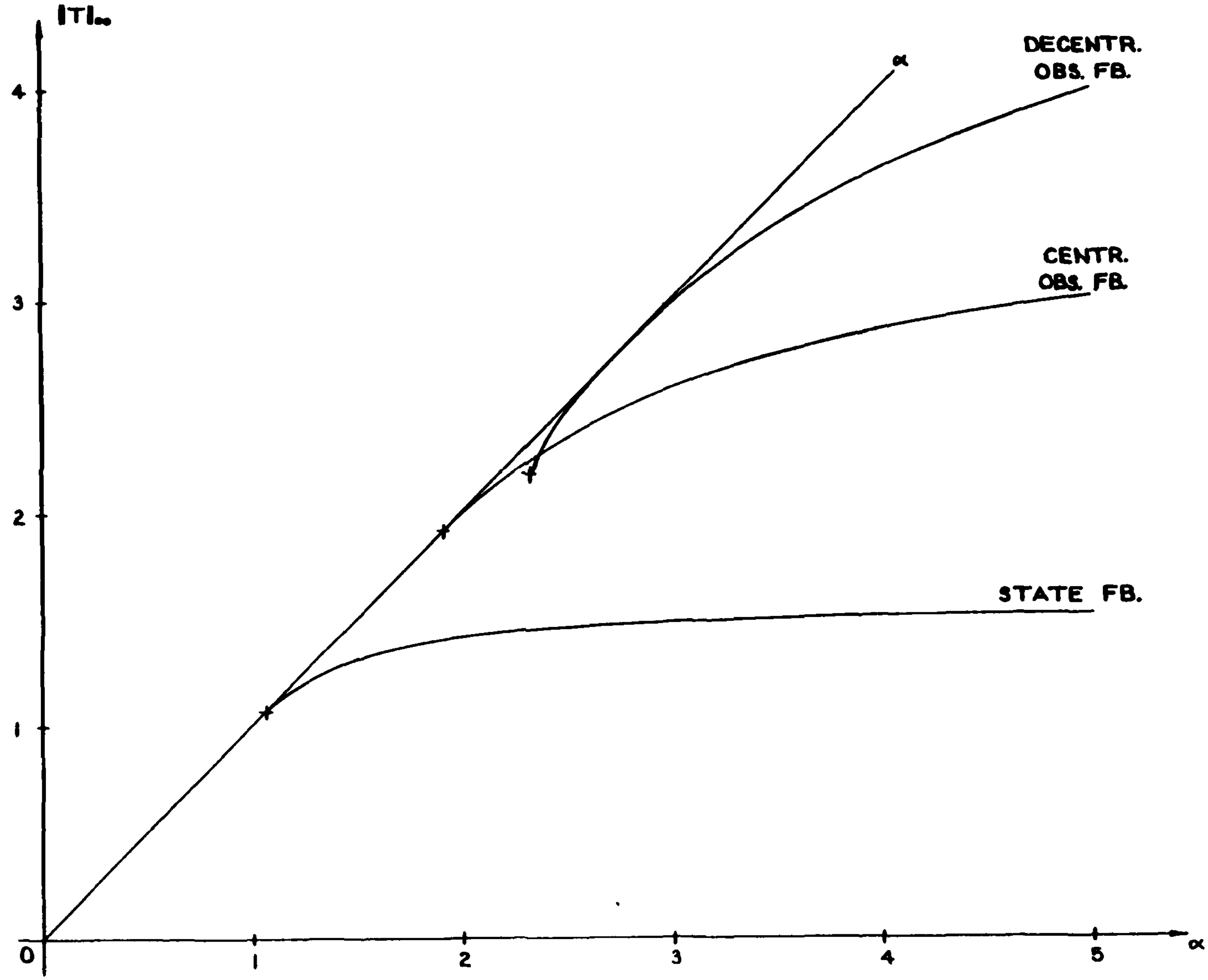


Figure 3.4: Comparison of actual closed-loop H_∞ norms, Example 2.

Chapter 4

Robust Decentralized Control

Consider again the decentralized plant (1.14), now including the structured plant uncertainty

$$A = A_0 + \sum_{k=1}^{r} G_k M_k H_k, \tag{4.1}$$

where A_0 is known, the G_k's and H_k's give the structure of the uncertainty, and each unknown constant matrix M_k satisfies

$$\lambda_{\max}\left\{M_k M_k^T\right\} < \sigma_k^2, \quad k \in \{1, 2, \ldots, r\}. \tag{4.2}$$

If each positive bound σ_k is sufficiently small, then the design equations to be derived for robust control will have appropriate solutions. The design developed in this chapter accounts for the uncertainty (4.1), and gives a robust decentralized control law for the plant. The decentralized results also apply easily to the simpler state-feedback and centralized output-feedback cases, which are omitted.

4.1 Robust Design Derivation

With no plant uncertainty assumed, the basic control law of Theorem 3.1 guarantees stability and the H_∞-norm bound $\|T\|_\infty \leq \alpha$ for the closed-loop system (3.9) by providing a solution $X_e \geq 0$ to the algebraic Riccati equation

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e + P_e = 0 \tag{4.3}$$

with $P_e = 0$, and (F_e, H_e) a detectable pair. Now suppose that, like the controller (3.8) of the basic design, the i^{th} controller has the form

$$\dot{\xi}_i = (A_{0\alpha} - L_i C_i)\xi_i + L_i y_i \tag{4.4a}$$

$$u_i = -B_i^T X \xi_i. \tag{4.4b}$$

Suppose also that L_i, $i \in \{1, 2, \ldots, q\}$, are chosen so that

$$F_{0e}^T X_e + X_e F_{0e} + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e + P_e = 0 \tag{4.5}$$

for some $P_e \geq 0$, where (F_{0e}, G_e, H_e) describes the nominal closed-loop system. In (4.4) and (4.5), plant uncertainty terms are omitted, so that

$$F_{0e} = \begin{pmatrix} A_0 & -BB_c^T X_c \\ L_c C & A_{0\alpha c} - L_c C_c \end{pmatrix},$$

where $A_{0\alpha c} = \mathrm{Diag}(A_{0\alpha}, A_{0\alpha}, \ldots, A_{0\alpha})$. We now proceed to determine a choice of $P_e \geq 0$ in (4.5) which will guarantee

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e \leq 0 \tag{4.6}$$

where

$$F_e = F_{0e} + \sum_{k=1}^{r} \begin{pmatrix} G_k \\ 0 \end{pmatrix} M_k (H_k \ \ 0) \equiv F_{0e} + \sum_{k=1}^{r} G_{ek} M_k H_{ek}. \tag{4.7}$$

Assuming (4.5) holds, then

$$\begin{aligned} F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e &= -P_e + (F_e - F_{0e})^T X_e + X_e (F_e - F_{0e}) \\ &= -P_e + \sum_{k=1}^{r} \left\{ H_{ek}^T M_k^T G_{ek}^T X_e + X_e G_{ek} M_k H_{ek} \right\}. \end{aligned} \tag{4.8}$$

Recall that σ_k is given by (4.2), and set

$$P_e = \sum_{k=1}^{r} \left\{ H_{ek}^T H_{ek} + \sigma_k^2 X_e G_{ek} G_{ek}^T X_e \right\} \geq \sum_{k=1}^{r} \left\{ H_{ek}^T H_{ek} + X_e G_{ek} M_k M_k^T G_{ek}^T X_e \right\}, \tag{4.9}$$

so that (4.8) gives

$$
\begin{aligned}
&F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e \\
&= \sum_{k=1}^{r} \left\{ H_{ek}^T M_k^T G_{ek}^T X_e + X_e G_{ek} M_k H_{ek} - H_{ek}^T H_{ek} - X_e G_{ek} M_k M_k^T G_{ek}^T X_e \right\} \\
&\quad - \sum_{k=1}^{r} \left\{ X_e G_{ek} (\sigma_k^2 I - M_k M_k^T) G_{ek}^T X_e \right\} \\
&= -\sum_{k=1}^{r} \left\{ H_{ek}^T - X_e G_{ek} M_k \right\} \left\{ H_{ek} - M_k^T G_{ek}^T X_e \right\} \\
&\quad - \sum_{k=1}^{r} \left\{ X_e G_{ek} \left(\sigma_k^2 I - M_k M_k^T \right) G_{ek}^T X_e \right\} \leq 0.
\end{aligned}
\tag{4.10}
$$

Therefore, if $X_e \geq 0$ satisfies (4.5), with P_e given by (4.9), then (4.6) holds, satisfying the main hypothesis of Lemma 2.1 for the uncertain system.

The next step in the derivation of the robust control is to determine the needed modifications to the design equations (3.3) and (3.19) so that $X_e \geq 0$ satisfies (4.5), with P_e given by (4.9). By examination of (4.5) and (4.9), and of the definitions of G_e and H_e given in (3.9), it is easily seen that

$$
F_{0e}^T X_e + X_e F_{0e} + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e + P_e = F_{0e}^T X_e + X_e F_{0e} + \frac{1}{\alpha^2} X_e G_e + G_{e+}^T X_e + H_{e+}^T H_{e+},
$$

where

$$
G_{e+} = \begin{pmatrix} G_{e+} & 0 \\ 0 & L_c \end{pmatrix}, \; G_+ = \begin{pmatrix} G & \alpha\sigma_1 G_1 & \dots & \alpha\sigma_r G_r \end{pmatrix}, \tag{4.11a}
$$

$$
H_{e+} = \begin{pmatrix} H_+ & 0 \\ 0 & -B_c^T X_c \end{pmatrix}, \; H_+ = \begin{pmatrix} H \\ H_1 \\ \vdots \\ H_r \end{pmatrix}. \tag{4.11b}
$$

Hence, the robust design is obtained by replacing the triple (A, G, H) with the triple (A_0, G_+, H_+) in the design equations (3.3) and (3.19) for the basic design. Using the augmented matrices G_+ and H_+ in the design equations is similar to introducing additional disturbance inputs and regulated outputs into the problem. Therefore, the smallest value of α for which the design equations will have a solution will be larger for the robust design than for the basic design.

Recall that, in the basic design, the controller dynamics depend on an assumed worst disturbance, and hence on the matrix G. Therefore, replacing G with G_+ in the design affects not only G_e, but also F_{0e}.

The final step in deriving the robust design is to establish that (F_e, H_e) is a detectable pair. Note that Lemma 3.1, applied to the modified design, establishes that (F_{0e}, H_{e+}) is a detectable pair, provided (A_0, H) is a detectable pair, $A_{0\alpha} \equiv A_0 + \alpha^{-2} G_+ G_+^T X - SX$ is Hurwitz, and $A_{0\alpha} + SX$ has no $j\omega$-axis eigenvalues. Let $v \neq 0$ satisfy

$$F_e v = \lambda v, \quad H_e v = 0. \tag{4.12}$$

The detectability proof consists of proving that $Re(\lambda) < 0$. Multiply (4.10) on the left by v^* and on the right by v to obtain

$$\begin{aligned} 2Re(\lambda) v^* X_e v + \frac{1}{\alpha^2} v^* X_e G_e G_e^T X_e v \quad &+ \quad \sum_{k=1}^{r} v^* \left\{ H_{ek}^T - X_e G_{ek} M_k \right\} \left\{ H_{ek} - M_k^T G_{ek}^T X_e \right\} v \\ &+ \quad \sum_{k=1}^{r} v^* \left\{ X_e G_{ek} \left(\sigma_k^2 I - M_k M_k^T \right) G_{ek}^T X_e \right\} v \leq 0. \end{aligned} \tag{4.13}$$

Since every term in (4.13) but the first is nonnegative, this implies

$$Re(\lambda) v^* X_e v \leq 0. \tag{4.14}$$

If inequality holds in (4.14), then $v^* X_e v > 0$ and $Re(\lambda) < 0$. If equality holds, then every term in (4.13) is zero. This gives

$$\left\{ H_{ek} - M_k^T G_{ek}^T X_e \right\} v = 0, \quad k \in \{1, 2, \ldots, r\}. \tag{4.15}$$

But, since (4.2) implies that $\sigma_k^2 I - M_k M_k^T$ is nonsingular, (4.13) also gives $G_{ek}^T X_e v = 0$; therefore, (4.15) gives

$$H_{ek} v = 0, \quad k \in \{1, 2, \ldots, r\}. \tag{4.16}$$

Hence, (4.7) and (4.12) give $F_{0e} v = \lambda v$, while (4.11b) and (4.12) give $H_{e+}^T H_{e+} v = 0$. Since (F_{0e}, H_{e+}) is a detectable pair, this implies $Re(\lambda) < 0$.

Theorem 4.1 summarizes the result. The following definitions are convenient:

$$A_{0\alpha} = A_0 + \frac{1}{\alpha^2} G_+ G_+^T X - SX, \tag{4.17a}$$

$$A_{0\alpha c} = \text{Diag}\,(A_{0\alpha}, A_{0\alpha}, \ldots, A_{0\alpha}), \tag{4.17b}$$

$$A_{0c} = A_{0\alpha c} + I_c B B_c^T X_c, \tag{4.17c}$$

$$G_{c+} = I_c G_+. \tag{4.17d}$$

Recall that I_c is given by (3.10g).

Theorem 4.1. *Suppose the plant (1.14) has constant structured uncertainty (4.1), with*

$$\lambda_{\max}\left\{M_k M_k^T\right\} < \sigma_k, \quad k \in \{1, 2, \ldots, r\}.$$

Define G_+ and H_+ as in (4.11), and let $X \geq 0$ satisfy

$$A_0^T X + X A_0 + \frac{1}{\alpha^2} X G_+ G_+^T X - X S X + H_+^T H_+ = 0, \tag{4.18}$$

and $W > 0$ satisfy the Riccati-like algebraic equation

$$\begin{aligned} W A_{0c}^T + A_{0c} W + \frac{1}{\alpha^2} W X_c B_c B_c^T X_c W - W C_c^T C_c W \\ + G_{c+} G_{c+}^T + (W - W_D) C_c^T C_c (W - W_D) = 0. \end{aligned} \tag{4.19}$$

Suppose also that (A_0, H) is a detectable pair, $A_{0\alpha}$ is Hurwitz, and $A_{0\alpha} + SX$ has no eigenvalues on the $j\omega$-axis. Then the decentralized control law

$$\dot{\xi}_i = (A_{0\alpha} - L_i C_i)\xi_i + L_i y_i, \quad i \in \{1, 2, \ldots, q\},$$

$$u_i = -B_i^T X \xi_i, \quad i \in \{1, 2, \ldots, q\},$$

with $L_i = W_{ii} C_i^T$, $i \in \{1, 2, \ldots, q\}$, robustly stabilizes the uncertain plant, and the closed-loop transfer-function matrix $T(s)$ from w_e to z satisfies

$$\|T\|_\infty \leq \alpha.$$

There is no explicit restriction on the size of the bounds σ_k in Theorem 4.1. However, the larger the σ_k's are taken to be, the larger α will need to be to obtain solutions to the design equations (4.18) and (4.19); if the σ_k's are taken to be too large, no solutions may exist at all. If bounds σ_k on the size of the uncertainty are known accurately, then these bounds should be incorporated in G_+ (or H_+), and hence in the design equations. If the design equations can then be solved, then the design can tolerate uncertainties of the specified size. On the other hand, if uncertainty bounds are not accurately known, the choice of the σ_k's may be used to reflect a relative weighting to be

given by the design to disturbance attenuation and robustness considerations. Since changing the values of the σ_k's in this case is equivalent to rescaling the G_k's and H_k's while holding the σ_k's fixed, it may simplify the design procedure to set

$$\sigma_k = \frac{1}{\alpha}, \quad k \in \{1, 2, \ldots, r\},$$

and scale the G_k's and H_k's so as to reflect the tradeoff between robustness and disturbance attenuation. Then, the size of the uncertainty which may be tolerated is determined indirectly by finding the smallest value of α for which the design equations can be solved. This variation on the design of Theorem 4.1 is given in Theorem 4.2.

Theorem 4.2. *Suppose the plant (1.14) has constant structured uncertainty (4.1), with*

$$\lambda_{\max}\left\{M_k M_k^T\right\} < \frac{1}{\alpha^2}, \quad k \in \{1, 2, \ldots, r\}.$$

Define $G_+ = (G \;\; G_1 \;\; \ldots \;\; G_r)$ *and* $H_+^T = (H^T \;\; H_1^T \;\; \ldots \;\; H_r^T)$. *Let* $X \geq 0$ *satisfy (4.18) and let* $W > 0$ *satisfy the Riccati-like algebraic equation (4.19). Suppose also that* (A_0, H) *is a detectable pair,* $A_{0\alpha}$ *is Hurwitz, and* $A_{0\alpha} + SX$ *has no eigenvalues on the* $j\omega$*-axis. Then the decentralized control law*

$$\dot{\xi}_i = (A_{0\alpha} - L_i C_i)\xi_i + L_i y_i, \quad i \in \{1, 2, \ldots, q\},$$

$$u_i = -B_i^T X \xi_i, \quad i \in \{1, 2, \ldots, q\},$$

with $L_i = W_{ii} C_i^T$, $i \in \{1, 2, \ldots, q\}$, *robustly stabilizes the uncertain plant, and the closed-loop transfer-function matrix* $T(s)$ *from* w_e *to* z *satisfies*

$$\|T\|_\infty \leq \alpha.$$

4.2 Example

This section presents an example of robust state-feedback control design. The example illustrates the difference between the robust designs of Theorems 4.1 and 4.2, and the use of the parameter σ_1 to determine the largest uncertainty in a certain class for which the design guarantees stability and the predetermined H_∞-norm bound. For these purposes, the state-feedback example is adequate, and has the advantage of avoiding the complication of decentralized design, already studied in Chapter 3.

Consider the plant of Section 3.2, now in a state-feedback setting, where the matrices

$$A_0 = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 2 \\ -1 & 0 & -2 & -3 \\ -2 & -1 & 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad G = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad H = (1 \;\; 0 \;\; -1 \;\; 0)$$

describe the nominal plant. Introduce the structured uncertainty

$$A = A_0 + G_1 M_1 H_1,$$

where M_1 is an unknown scalar, and G_1 and H_1 are given by

$$G_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \; H_1 = (0 \;\; 0 \;\; 1 \;\; 0). \tag{4.20}$$

This represents an uncertainty in the (4,3) element of the A-matrix of the plant. As in the decentralized design of Theorem 4.1, the robust state-feedback control is found by doing a basic design, but with the augmented matrices G_+ and H_+ in place of G and H, where in this case

$$G_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \alpha\sigma_1 \end{pmatrix}, \; H_+ = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The state-feedback design equation becomes

$$A_0^T X + X A_0 + \frac{1}{\alpha^2} X G_+ G_+^T X - XSX + H_+^T H_+ = 0,$$

or equivalently

$$A_0^T X + X A_0 + \frac{1}{\alpha^2} X G G^T X + \sigma_1^2 X G_1 G_1^T X - XSX + H^T H + H_1^T H_1 = 0. \tag{4.21}$$

In the second quadratic term of (4.21), the α's cancel out, allowing computation of a solution for $\alpha = \infty$. By setting $\alpha = \infty$ and solving (4.21) with various values of σ_1, one may determine a largest plant perturbation (corresponding to $|M_1| = \sigma_{1_{max}}$) for which at least stability can be guaranteed

using the robust state-feedback design. Then, given any $\sigma_1 \leq \sigma_{1_{max}}$, one may determine a number $\alpha_{\min}$ such that for any $\alpha \geq \alpha_{\min}$ there exists an appropriate solution of (4.21), and therefore an associated design guaranteeing the robust H_∞-norm bound $\|T\|_\infty \leq \alpha$ for the closed-loop system. Table 4.1 gives, to the nearest 0.1, the values of $\alpha_{\min}$ computed for various values of σ_1, and shows clearly the tradeoff between robustness and optimal disturbance rejection. In this example, the largest admissible plant perturbation is given approximately by $\sigma_{1_{max}} = 1.8$.

If $\sigma_1 = \alpha^{-1}$, as in Theorem 4.2, then for G_1 and H_1 given by (4.20) the design equation becomes

$$A_0^T X + X A_0 + \frac{1}{\alpha^2} X \left(G G^T + G_1 G_1^T \right) X - X S X + H^T H + H_1^T H_1 = 0. \tag{4.22}$$

The approximate smallest value of α for which (4.22) has an appropriate solution is $\alpha_{\min} = 1.4$, which corresponds to a plant uncertainty bound $\sigma_1 = 0.71$.

Table 4.1: Approximate minimum H_∞-norm bounds for various plant uncertainties.

σ_1	1.0	1.2	1.4	1.6	1.8
$\alpha_{\min}$	1.4	1.5	1.6	1.8	3.0

Chapter 5

Reliable Control Design

This chapter develops reliable centralized and decentralized control designs which guarantee stability and H_∞ disturbance attenuation despite possible measurement or control failures. First is presented an example which establishes the desirability of a reliable decentralized design. Next, centralized reliable designs are presented which guarantee stability and an H_∞-norm bound despite possible outages of sensors or actuators within predefined susceptible sets. The cases of sensor and actuator outages are treated separately, resulting in two designs with different reliability properties. Then, decentralized reliable designs are presented which guarantee stability and an H_∞-norm bound despite possible outages of whole controllers within a predefined set of susceptible controllers. The controller outages are modelled first as measurement outages, and then as control input outages, resulting in two distinct designs with the same reliability properties.

5.1 Motivation

The 4^{th}-order example of Section 3.2 is used to motivate the development of a reliable decentralized control. In this example, stability and a predetermined H_∞-norm bound are guaranteed by the basic decentralized design for various values of the design parameter α. Table 5.1 gives the actual H_∞ norms of the closed-loop systems corresponding to several values of α. In addition to the case when no controller failure occurs, Table 5.1 gives the conditions corresponding to a failure of each of the two controllers. A failure of Controller #1 results in instability for each design computed, while a failure of Controller #2 results only in an increased H_∞ norm for the closed-loop system.

Table 5.1. H_∞ norms for the basic decentralized design.

	no failure	#1 fails	#2 fails
$\alpha = 20$	3.64	unstable	5.34
$\alpha = 16$	3.63	unstable	5.30
$\alpha = 14$	3.61	unstable	5.28
$\alpha = 12$	3.59	unstable	5.23
$\alpha = 8$	3.49	unstable	5.04
$\alpha = 4$	3.05	unstable	4.19
$\alpha = 2$	1.995	unstable	2.46

Since the plant is open-loop unstable, a failure of both controllers at once necessarily results in instability; however, it would be desirable to alter the design so as to guarantee at least stability, and, better still, some level of disturbance attenuation for the closed-loop system if only one controller should fail. While the basic design in this case still works well if only Controller #2 fails, it is not acceptable if Controller #1 fails. Therefore, a design reliable with respect to failure of Controller #1 is desired.

The approach to reliable design developed here is similar to that for robust design developed in Chapter 4. The essential idea is that, if there exists $X_e \geq 0$ satisfying

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_e^T H_e + P_e = 0 \tag{5.1}$$

with some $P_e \geq 0$, then the resulting closed-loop system will by Lemma 2.1 be stable and have H_∞-norm bound α. By judicious choice of P_e, additional system properties associated with reliability can be assured. Then, perturbations are introduced into the basic design equations such that (5.1) is satisfied for that choice of P_e. As in the robust design, the appropriate choices of P_e are equivalent to appending columns or rows to G or H in the basic design equations.

5.2 Reliable Centralized Design

The problem addressed here is that of designing a centralized controller which is reliable despite possible sensor or actuator outages. The outages will be restricted to occur within a preselected subset of available measurements or control inputs. The controllers developed will guarantee closed-loop stability and a predetermined H_∞-norm bound, regardless of admissible sensor or actuator failures. The cases of sensor and actuator outages are treated separately, and two designs are

developed to handle the two cases. However, it will be clear from the results that controllers which can handle both sensor and actuator outages can be obtained by combining the designs.

Consider first the design of a controller that can tolerate the outage of certain sensors which provide the various elements of the measurement vector y. Let $\Omega \subseteq \{1, 2, \ldots, \dim(y)\}$ correspond to a selected subset of sensors susceptible to outages. Introduce the decomposition

$$C = C_\Omega + C_{\bar{\Omega}}, \tag{5.2}$$

where C_Ω denotes the measurement matrix associated with Ω, and $C_{\bar{\Omega}}$ denotes the measurement matrix associated with the complementary subset of measurements. In other words, $C_{\bar{\Omega}}$ is the same as C, but with rows corresponding to susceptible sensors zeroed out. Let $\omega \subseteq \Omega$ correspond to a particular subset of the susceptible sensors that actually experience an outage, and let $T_{\bar{\omega}}(s)$ denote the transfer-function matrix of the resulting closed-loop system. It is convenient to adopt the notation

$$C = C_\omega + C_{\bar{\omega}} \tag{5.3}$$

where C_ω and $C_{\bar{\omega}}$ have meanings analogous to those of C_Ω and $C_{\bar{\Omega}}$ in (5.2). Since $\omega \subseteq \Omega$, $C_\omega^T C_\omega \leq C_\Omega^T C_\Omega$. Also decompose the observer gain as

$$L = L_\omega + L_{\bar{\omega}} \tag{5.4}$$

so that

$$LC = L_\omega C_\omega + L_{\bar{\omega}} C_{\bar{\omega}}.$$

($L_{\bar{\omega}}$ has columns zeroed out corresponding to sensors which have actually failed.) Then the following result holds:

Theorem 5.1. *With all assumptions and the design otherwise as in Theorem 2.1, assume $X \geq 0$ and $Y > 0$ satisfy the AREs*

$$A^T X + XA - XSX + \frac{1}{\alpha^2} XGG^T X + H^T H + \alpha^2 C_\Omega^T C_\Omega = 0, \tag{5.5}$$

$$AY + YA^T + \frac{1}{\alpha^2} YH^T HY - YC_\Omega^T C_\Omega Y + GG^T = 0, \tag{5.6}$$

respectively. Then, for sensor outages corresponding to any $\omega \subseteq \Omega$, the closed-loop system is stable, and $\|T_{\bar{\omega}}\|_\infty \leq \alpha$.

Remark: With all sensors operational, corresponding to $\omega = \emptyset$, $T_{\bar{\omega}}(s) = T(s)$ is the transfer-function matrix from w_e to z, where

$$w_e = \begin{pmatrix} w_0 \\ w \end{pmatrix}, \quad z = \begin{pmatrix} Hx \\ u \end{pmatrix}.$$

Theorem 5.1 covers this case automatically, since $w = \emptyset \subseteq \Omega$. If sensors corresponding to a nonempty subset $\omega \subseteq \Omega$ fail, then $T_{\bar{\omega}}(s)$ is the transfer-function matrix from $w_{e\bar{\omega}}$ to z, where

$$w_{e\bar{\omega}} = \begin{pmatrix} w_0 \\ w_{\bar{\omega}} \end{pmatrix},$$

with $w_{\bar{\omega}}$ containing only those components of measurement noise associated with operational sensors.

Proof: The design equations (5.5) and (5.6) arise from replacing H in the description of the plant by the augmented matrix

$$H_+ = \begin{pmatrix} H \\ \alpha C_\Omega \end{pmatrix}, \tag{5.7}$$

and changing the design equations accordingly. If (5.5) and (5.6) have appropriate solutions, then Theorem 2.1 guarantees that $X_e \geq 0$ satisfies

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_{e+}^T H_{e+} = 0, \tag{5.8}$$

where the augmented closed-loop system is described by the matrices

$$F_e = \begin{pmatrix} A & -SX \\ LC & A_\alpha - LC \end{pmatrix}, \; G_e = \begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix}, \; H_{e+} = \begin{pmatrix} H_+ & 0 \\ 0 & -B^T X \end{pmatrix}, \tag{5.9}$$

and (F_e, H_{e+}) is a detectable pair. The actual closed-loop system with no sensor outages is described by the matrices

$$F_e = \begin{pmatrix} A & -SX \\ LC & A_\alpha - LC \end{pmatrix}, \; G_e = \begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix}, \; H_e = \begin{pmatrix} H & 0 \\ 0 & -B^T X \end{pmatrix}. \tag{5.10}$$

For sensor outages corresponding to $\omega \subseteq \Omega$, the controller becomes

$$\dot{\xi} = (A + \frac{1}{\alpha^2} G G^T X - SX - LC)\xi + L_{\bar{\omega}} y, \tag{5.11a}$$

$$u = -B^T X \xi. \tag{5.11b}$$

The controller dynamic structure is not affected by a sensor outage; only the controller input structure is effectively changed. Given (5.11), the closed-loop system matrices become

$$F_{e\bar{\omega}} = \begin{pmatrix} A & -SX \\ L_{\bar{\omega}}C_{\bar{\omega}} & A_\alpha - LC \end{pmatrix},\ G_{e\bar{\omega}} = \begin{pmatrix} G & 0 \\ 0 & L_{\bar{\omega}} \end{pmatrix},\ H_e = \begin{pmatrix} H & 0 \\ 0 & -B^T X \end{pmatrix}. \tag{5.12}$$

The following useful relations are derived from (5.9), (5.10), and (5.12):

$$F_e = F_{e\bar{\omega}} + \begin{pmatrix} 0 \\ L_\omega \end{pmatrix}(C_\omega\ 0) \equiv F_{e\bar{\omega}} + L_{e\omega}C_{e\omega}, \tag{5.13a}$$

$$G_eG_e^T = \begin{pmatrix} GG^T & 0 \\ 0 & L_{\bar{\omega}}L_{\bar{\omega}}^T \end{pmatrix} + \begin{pmatrix} 0 \\ L_\omega \end{pmatrix}(0\ L_\omega^T) = G_{e\bar{\omega}}G_{e\bar{\omega}}^T + L_{e\omega}L_{e\omega}^T, \tag{5.13b}$$

$$H_{e+}^TH_{e+} = H_e^TH_e + \alpha^2 \begin{pmatrix} C_\Omega^T C_\Omega & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.13c}$$

Use (5.8) and (5.13) to obtain

$$\begin{aligned} &F_{e\bar{\omega}}^T X_e + X_eF_{e\bar{\omega}} + \tfrac{1}{\alpha^2}X_eG_{e\bar{\omega}}G_{e\bar{\omega}}^TX_e + H_e^TH_e \\ &\quad = -C_{e\omega}^TL_{e\omega}^TX_e - X_eL_{e\omega}C_{e\omega} - \tfrac{1}{\alpha^2}X_eL_{e\omega}L_{e\omega}^TX_e - \alpha^2\left(\begin{smallmatrix} C_\Omega^T \\ 0 \end{smallmatrix}\right)(C_\Omega\ 0). \end{aligned} \tag{5.14}$$

Therefore, since $-C_\Omega^TC_\Omega \le -C_\omega^TC_\omega$, (5.14) gives

$$\begin{aligned} F_{e\bar{\omega}}^TX_e &+ X_eF_{e\bar{\omega}} + \tfrac{1}{\alpha^2}X_eG_{e\bar{\omega}}G_{e\bar{\omega}}^TX_e + H_e^TH_e \\ &\le -C_{e\omega}^TL_{e\omega}^TX_e - X_eL_{e\omega}C_{e\omega} - \tfrac{1}{\alpha^2}X_eL_{e\omega}L_{e\omega}^TX_e - \alpha^2C_{e\omega}^TC_{e\omega} \\ &= -\left(\tfrac{1}{\alpha}X_eL_{e\omega} + \alpha C_{e\omega}^T\right)\left(\tfrac{1}{\alpha}L_{e\omega}^TX_e + \alpha C_{e\omega}\right) \le 0. \end{aligned} \tag{5.15}$$

Hence, provided $(F_{e\bar{\omega}}, H_e)$ is a detectable pair, Lemma 2.1 guarantees that $F_{e\bar{\omega}}$ is Hurwitz, and that $T_{\bar{\omega}}(s) = H_e(sI - F_{e\bar{\omega}})^{-1}G_{e\bar{\omega}}$, the transfer-function matrix from $w_{e\bar{\omega}}$ to $z_{\bar{\omega}}$, satisfies $\|T_{\bar{\omega}}\|_\infty \le \alpha$. The detectability proof is routine: If $v^T = (v_1^T\ v_2^T) \ne 0$ satisfies $F_{e\bar{\omega}}v = \lambda v$ and $H_ev = 0$, then $Av_1 = \lambda v_1$ and $Hv_1 = 0$, with (A, H) assumed a detectable pair. Therefore, either $Re(\lambda) < 0$ or $v_1 = 0$. Suppose $v_1 = 0$; then $F_ev = F_{e\bar{\omega}}v = \lambda v$ and $H_ev = 0$ gives $H_{e+}v = 0$. Since (F_e, H_{e+}) is a detectable pair, $Re(\lambda) < 0$. Q.E.D.

Consider now the design of a controller that can tolerate the outage of certain actuators which provide the various elements of the control vector u. Let $\Omega \subseteq \{1, 2, ..., \dim(u)\}$ correspond to a selected subset of actuators susceptible to outages. Introduce the decomposition

$$B = B_\Omega + B_{\bar{\Omega}}, \tag{5.16}$$

where B_Ω denotes the control matrix associated with the set Ω, and $B_{\bar{\Omega}}$ denotes the control matrix associated with the complementary subset of control inputs. In other words, $B_{\bar{\Omega}}$ is the same as B, but with columns corresponding to susceptible actuators zeroed out. Let $\omega \subseteq \Omega$ correspond to a particular subset of the susceptible actuators that actually fail, and let $T_{\bar{\omega}}(s)$ denote the transfer-function matrix of the resulting closed-loop system. It is convenient to adopt the notation

$$B = B_\omega + B_{\bar{\omega}} \tag{5.17}$$

where B_ω and $B_{\bar{\omega}}$ have meanings analogous to those of B_Ω and $B_{\bar{\Omega}}$ in (5.16). Since $\omega \subseteq \Omega$, $B_\omega B_\omega^T \leq B_\Omega B_\Omega^T$. Then the following result, dual to Theorem 5.1, holds:

Theorem 5.2. *With all assumptions and the design otherwise as in Theorem 2.1, assume $X \geq 0$ and $Y > 0$ satisfy the AREs*

$$A^T X + XA - XB_{\bar{\Omega}} B_{\bar{\Omega}}^T X + \frac{1}{\alpha^2} XGG^T X + H^T H = 0, \tag{5.18}$$

$$AY + YA^T + \frac{1}{\alpha^2} YH^T HY - YC^T CY + GG^T + \alpha^2 B_\Omega B_\Omega^T = 0, \tag{5.19}$$

respectively. Define

$$G_+ = (G \;\; \alpha B_\Omega), \tag{5.20}$$

and let the controller be given by

$$\dot{\xi} = (A + \frac{1}{\alpha^2} G_+ G_+^T X - SX - LC)\xi + Ly, \tag{5.21a}$$

$$u = -B^T X \xi. \tag{5.21b}$$

Assume the controller is open-loop (internally) stable. Then, for actuator outages corresponding to any $\omega \subseteq \Omega$, the closed-loop system is stable, and $\|T_{\bar{\omega}}\|_\infty \leq \alpha$.

Remark: For actuator outages corresponding to $\omega \subseteq \Omega$, $T_{\bar{\omega}}(s)$ is the transfer-function matrix from w_e to $z_{\bar{\omega}}$, where $z_{\bar{\omega}}$ excludes control components associated with failed actuators.

Proof: The design equations (5.18) and (5.19) arise from replacing the matrix G in the description of the plant (1.3) with the augmented matrix G_+, and introducing the corresponding changes in the design equations. If (5.18) and (5.19) have appropriate solutions, then Theorem 2.1 guarantees that $X_e \geq 0$ satisfies

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_{e+} G_{e+}^T X_e + H_e^T H_e = 0, \tag{5.22}$$

where the augmented closed-loop system is described by the matrices

$$F_e = \begin{pmatrix} A & -SX \\ LC & A_\alpha - LC \end{pmatrix}, \; G_{e+} = \begin{pmatrix} G_+ & 0 \\ 0 & L \end{pmatrix}, \; H_e = \begin{pmatrix} H & 0 \\ 0 & -B^T X \end{pmatrix}, \tag{5.23}$$

with $A_\alpha \equiv A + \alpha^{-2} G_+ G_+^T X - SX$ and (F_e, H_e) a detectable pair. When there are no actuator outages, the actual closed-loop system is described by the matrices

$$F_e = \begin{pmatrix} A & -SX \\ LC & A_\alpha - LC \end{pmatrix}, \; G_e = \begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix}, \; H_e = \begin{pmatrix} H & 0 \\ 0 & -B^T X \end{pmatrix}. \tag{5.24}$$

For actuator outages corresponding to $\omega \subseteq \Omega$, the controller becomes

$$\dot{\xi} = \left(A + \frac{1}{\alpha^2} G_+ G_+^T X - SX - LC\right)\xi + Ly, \tag{5.25a}$$

$$u = -B_{\bar{\omega}}^T X \xi. \tag{5.25b}$$

The controller dynamic structure is not affected by actuator outages; only the controller output structure is effectively changed. Given (5.25), the closed-loop system is described by the matrices

$$F_{e\bar{\omega}} = \begin{pmatrix} A & -B_{\bar{\omega}} B_{\bar{\omega}}^T X \\ LC & A_\alpha - LC \end{pmatrix}, \; G_e = \begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix}, \; H_{e\bar{\omega}} = \begin{pmatrix} H & 0 \\ 0 & -B_{\bar{\omega}}^T X \end{pmatrix}. \tag{5.26}$$

The following useful relations are derived from (5.23), (5.24), and (5.26):

$$F_e = F_{e\bar{\omega}} - \begin{pmatrix} B_\omega \\ 0 \end{pmatrix} (0 \;\, B_\omega^T X) \equiv F_{e\bar{\omega}} - B_{e\omega}(0 \;\, B_\omega^T X), \tag{5.27a}$$

$$H_e^T H_e = H_{e\bar{\omega}}^T H_{e\bar{\omega}} + \begin{pmatrix} 0 \\ X B_\omega \end{pmatrix} (0 \;\, B_\omega^T X), \tag{5.27b}$$

$$G_{e+} G_{e+}^T = G_e G_e^T + \alpha^2 \begin{pmatrix} B_\Omega B_\Omega^T & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.27c}$$

Use (5.22) and (5.27) to obtain

$$\begin{aligned} F_{e\bar{\omega}}^T X_e \;&+\; X_e F_{e\bar{\omega}} + \tfrac{1}{\alpha^2} X_e G_e G_e^T X_e + H_{e\bar{\omega}}^T H_{e\bar{\omega}} \\ &\le\; -\left(X_e B_{e\omega} - \left(\begin{smallmatrix} 0 \\ X B_\omega \end{smallmatrix}\right)\right)\left(B_{e\omega}^T X_e - \left(0 \;\, B_\omega^T X\right)\right) \le 0. \end{aligned} \tag{5.28}$$

Provided $(F_{e\bar{\omega}}, H_{e\bar{\omega}})$ is a detectable pair, Lemma 2.1 guarantees that $F_{e\bar{\omega}}$ is Hurwitz, and that $T_{\bar{\omega}}(s) = H_{e\bar{\omega}}(sI - F_{e\bar{\omega}})^{-1} G_e$ satisfies $\|T_{\bar{\omega}}\|_\infty \le \alpha$. To prove detectability, let $v^T = (v_1^T \; v_2^T) \neq 0$

satisfy $F_{e\bar{\omega}}v = \lambda v$ and $H_{e\bar{\omega}}v = 0$; then $Av_1 = \lambda v_1$ and $Hv_1 = 0$, with (A, H) assumed a detectable pair. Therefore, either $Re(\lambda) < 0$ or $v_1 = 0$. If $v_1 = 0$, then $F_{e\bar{\omega}}v = \lambda v$ gives

$$(A + \frac{1}{\alpha^2}G_+G_+^TX - SX - LC)v_2 = \lambda v_2. \tag{5.29}$$

By the assumption that the controller is open-loop stable, $(A+\alpha^{-2}G_+G_+^TX - SX - LC)$ is Hurwitz; therefore, $Re(\lambda) < 0$. Q.E.D.

The design given in Theorem 5.2, unlike that given in Theorem 5.1, requires that the controller turn out stable in order to guarantee reliable closed-loop stability. If the design does not result in a stable controller, it may be combined with a strongly stabilizing design developed in Section 5.5; then the assumption of open-loop stability of the controller will hold automatically.

Note that to achieve reliability with respect to sensor outages, it is sufficient to modify the feedback and observer gains; however, to achieve reliability with respect to actuator outages, the observer structure must also be modified. The structural modification required is the inclusion of G_+ in the controller dynamic matrix.

5.3 Reliable Decentralized Design

Let $\Omega \subseteq \{1, 2, \ldots, q\}$ correspond to a subset of controllers subject to outages. The problem is to compute a decentralized control law which guarantees closed-loop stability and an H_∞-norm bound in spite of controller outages corresponding to any subset $\omega \subseteq \Omega$. Without loss of generality, $\Omega = \{t+1, t+2, \ldots, q\}$ and $\omega = \{r+1, r+2, \ldots, q\}$, with $r \geq t$. Introduce the decompositions

$$B = (B_1 \ldots B_r \;\; 0 \ldots 0) + (0 \ldots 0 \;\; B_{r+1} \ldots B_q) \equiv B_{\bar{\omega}} + B_\omega, \tag{5.30a}$$

$$B_c = \mathrm{Diag}(B_1, \ldots, B_r, 0, \ldots, 0) + \mathrm{Diag}(0, \ldots, 0, B_{r+1}, \ldots, B_q) \equiv B_{c\bar{\omega}} + B_{c\omega}, \tag{5.30b}$$

$$C = \begin{pmatrix} C_1 \\ \vdots \\ C_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ C_{r+1} \\ \vdots \\ C_q \end{pmatrix} \equiv C_{\bar{\omega}} + C_\omega, \tag{5.30c}$$

$$C_c = \mathrm{Diag}(C_1, \ldots, C_r, 0, \ldots, 0) + \mathrm{Diag}(0, \ldots, 0, C_{r+1}, \ldots, C_q) \equiv C_{c\bar{\omega}} + C_{c\omega}, \tag{5.30d}$$

$$L_c = \mathrm{Diag}(L_1, \ldots, L_r, 0, \ldots, 0) + \mathrm{Diag}(0, \ldots, 0, L_{r+1}, \ldots, L_q) \equiv L_{c\bar{\omega}} + L_{c\omega}. \tag{5.30e}$$

Also decompose the disturbance and regulated output vectors as

$$w_e = \begin{pmatrix} w_0 \\ w \end{pmatrix} = \begin{pmatrix} w_0 \\ w_{\bar{\omega}} \\ w_\omega \end{pmatrix} = \begin{pmatrix} w_{e\bar{\omega}} \\ w_\omega \end{pmatrix}, \tag{5.31a}$$

$$z = \begin{pmatrix} Hx \\ u_{\bar{\omega}} \\ u_\omega \end{pmatrix} = \begin{pmatrix} z_{\bar{\omega}} \\ u_\omega \end{pmatrix}. \tag{5.31b}$$

Finally, define

$$B_\Omega = (B_{t+1} \ldots B_q), \tag{5.32a}$$

$$C_\Omega^T = (C_{t+1}^T \ldots C_q^T). \tag{5.32b}$$

Note that for any $\omega \subseteq \Omega$,

$$B_\Omega B_\Omega^T \geq B_\omega B_\omega^T, \tag{5.33}$$

$$C_\Omega^T C_\Omega \geq C_\omega^T C_\omega. \tag{5.34}$$

When no controller failures occur, the closed-loop system is described by matrices of the form

$$F_e = \begin{pmatrix} A & -BB_c^T \\ L_c C & A_{\alpha c} - L_c C_c \end{pmatrix}, \; G_e = \begin{pmatrix} G & 0 \\ 0 & L_c \end{pmatrix}, \; H_e = \begin{pmatrix} H & 0 \\ 0 & -B_c^T X_c \end{pmatrix},$$

where $A_{\alpha c} = \mathrm{Diag}(A_\alpha, A_\alpha, \ldots, A_\alpha)$. Suppose that controller failures take the form

$$y_i = 0, \quad i \in \omega. \tag{5.35}$$

The closed-loop system then takes the form

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & -BB^T X_c \\ L_{c\bar{\omega}} C_{\bar{\omega}} & A_{\alpha c} - L_c C_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & L_{c\bar{\omega}} \end{pmatrix} \begin{pmatrix} w_0 \\ w \end{pmatrix} = F_{e\bar{\omega}} x_e + G_{e\bar{\omega}} w_e, \tag{5.36a}$$

$$z = \begin{pmatrix} H & 0 \\ 0 & -B_c^T X_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = H_e x_e \tag{5.36b}$$

Because of the assumed mode of failure, given by (5.35), the disturbances w_i, $i \in \omega$, do not enter the system (5.36). In fact, (5.36) is a controllability canonical form, with ξ_i, $i \in \omega$, the uncontrollable parts of the extended state vector. Note also that

$$A_{\alpha c} - L_c C_c = \mathrm{Diag}(A_\alpha - L_1 C_1, A_\alpha - L_2 C_2, \ldots, A_\alpha - L_q C_q), \tag{5.37}$$

where $A_\alpha - L_i C_i$ is the open-loop dynamic matrix of the i^{th} controller. Because of the form of (5.36), the open-loop eigenvalues of the controllers which have failed appear directly as modes of the closed-loop system. This means that a design guaranteeing reliable stability will automatically guarantee that all controllers susceptible to outages are open-loop stable.

It is convenient to note that $F_{e\bar{\omega}}$ and $G_{e\bar{\omega}}$ are related to F_e and G_e by

$$F_{e\bar{\omega}} = F_e - \begin{pmatrix} 0 \\ L_{c\omega} \end{pmatrix} (C_\omega \quad 0) \equiv F_e - L_{e\omega} C_{e\omega}, \tag{5.38a}$$

$$G_{e\bar{\omega}} = G_e - \begin{pmatrix} 0 & 0 \\ 0 & L_{c\omega} \end{pmatrix}, \tag{5.38b}$$

$$G_{e\bar{\omega}} G_{e\bar{\omega}}^T = G_e G_e^T - L_{e\omega} L_{e\omega}^T. \tag{5.38c}$$

The design which follows will guarantee that $F_{e\bar{\omega}}$ is Hurwitz, and that the transfer-function matrix $T_{\bar{\omega}}(s) = H_e(sI - F_{e\bar{\omega}})^{-1} G_{e\bar{\omega}}$ satisfies $\|T_{\bar{\omega}}\|_\infty \le \alpha$, for controller outages associated with any $\omega \subseteq \Omega$. The case where no controllers fail (represented by $\omega = \emptyset \subseteq \Omega$) is always admissible; hence, the design will automatically guarantee that F_e is Hurwitz and that $T(s) = H_e(sI - F_e)^{-1} G_e$ satisfies $\|T\|_\infty \le \alpha$. The following theorem gives the reliable design method.

Theorem 5.3. *With all assumptions and the decentralized design otherwise as in Theorem 3.1, let $X \ge 0$ satisfy*

$$A^T X + X A + \frac{1}{\alpha^2} X G G^T X - X B B^T X + H^T H + \alpha^2 C_\Omega^T C_\Omega = 0, \tag{5.39}$$

where $\Omega \subseteq \{1, 2, \ldots, q\}$. Then, for controller outages corresponding to any $\omega \subseteq \Omega$, the closed-loop system (5.8) is internally stable, and the closed-loop transfer-function matrix $T_{\bar{\omega}}(s)$ from $w_{e\bar{\omega}}$ to $z_{\bar{\omega}}$ satisfies $\|T_{\bar{\omega}}\|_\infty \le \alpha$. In addition, all controllers corresponding to the "susceptible" set Ω are open-loop stable.

Remark: The design given in Theorem 5.3 results from replacing H in the description of the plant (1.14) with the augmented matrix

$$H_+ = \begin{pmatrix} H \\ \alpha C_\Omega \end{pmatrix}, \tag{5.40}$$

and changing the design equations accordingly. This substitution results in no change in the design equation (3.19), and is equivalent to selecting P_e in (5.1) as

$$P_e = \begin{pmatrix} \alpha^2 C_\Omega^T C_\Omega & 0 \\ 0 & 0 \end{pmatrix} \geq 0. \tag{5.41}$$

The basic decentralized design computed for the augmented plant will provide reliable control for the actual plant.

Proof: Just as in the development of Section 3.1, the existence of appropriate solutions to the perturbed design equations (5.39) and (3.19) guarantees that $X_e \geq 0$ satisfies

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_e G_e^T X_e + H_{e+}^T H_{e+} = 0, \tag{5.42}$$

where

$$H_{e+} = \begin{pmatrix} H_+ & 0 \\ 0 & -B_c^T X_c \end{pmatrix}. \tag{5.43}$$

Now (5.38), (5.40), (5.42), and (5.43) give

$$\begin{aligned} F_{e\bar{\omega}}^T X_e &+ X_e F_{e\bar{\omega}} + \tfrac{1}{\alpha^2} X_e G_{e\bar{\omega}} G_{e\bar{\omega}}^T X_e + H_e^T H_e \\ &= -C_{e\omega}^T L_{e\omega}^T X_e - X_e L_{e\omega} C_{e\omega} - \tfrac{1}{\alpha^2} X_e L_{e\omega} L_{e\omega}^T X_e - \alpha^2 \left(\begin{smallmatrix} C_\Omega^T \\ 0 \end{smallmatrix}\right)(C_\Omega \ 0). \end{aligned}$$

Therefore, by (5.34),

$$\begin{aligned} F_{e\bar{\omega}}^T X_e &+ X_e F_{e\bar{\omega}} + \tfrac{1}{\alpha^2} X_e G_{e\bar{\omega}} G_{e\bar{\omega}}^T X_e + H_e^T H_e \\ &\leq -C_{e\omega}^T L_{e\omega}^T X_e - X_e L_{e\omega} C_{e\omega} - \tfrac{1}{\alpha^2} X_e L_{e\omega} L_{e\omega}^T X_e - \alpha^2 C_{e\omega}^T C_{e\omega} \\ &= -\left(\tfrac{1}{\alpha} X_e L_{e\omega} + \alpha C_{e\omega}^T\right)\left(\tfrac{1}{\alpha} L_{e\omega}^T X_e + \alpha C_{e\omega}\right) \leq 0. \end{aligned}$$

Hence, provided $(F_{e\bar{\omega}}, H_e)$ is a detectable pair, Lemma 2.1 guarantees that $F_{e\bar{\omega}}$ is Hurwitz, and that $T_{\bar{\omega}}(s) = H_e(Is - F_{e\bar{\omega}})^{-1} G_{e\bar{\omega}}$, the transfer-function matrix from $w_{e\bar{\omega}}$ to $z_{\bar{\omega}}$, satisfies $\|T_{\bar{\omega}}\|_\infty \leq \alpha$. The detectability proof is the same as that of Lemma 3.1: Assuming $v \neq 0$ satisfies $F_{e\bar{\omega}} v = \lambda v$ and $H_e v = 0$ gives $A v_1 = \lambda v_1$ and $H v_1 = 0$, with (A, H) assumed a detectable pair. Therefore, either $Re(\lambda) < 0$ or $v_1 = 0$. If $v_1 = 0$, then $(A_{\alpha c} - L_c C_c) v_2 = \lambda v_2$, and hence $(A_c - L_c C_c) v_2 = \lambda v_2$, where $A_c - L_c C_c$ is known to be Hurwitz. Q.E.D.

Recall that the closed-loop system (5.36) assumes measurement failures corresponding to each $i \in \omega$. If instead there are control input failures, that is, if the controller failures are given by

$$u_i = 0, \quad i \in \omega, \tag{5.44}$$

then the closed-loop system has the form

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & -B_{\bar{\omega}} B_{c\bar{\omega}}^T X_c \\ L_c C & A_{\alpha c} - L_c C_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & L_c \end{pmatrix} \begin{pmatrix} w_0 \\ w \end{pmatrix} \equiv F_{e\bar{\omega}} x_e + G_e w_e \tag{5.45a}$$

$$z = \begin{pmatrix} H & 0 \\ 0 & -B_{c\bar{\omega}}^T X_c \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \equiv H_{e\bar{\omega}} x_e, \tag{5.45b}$$

where $F_{e\bar{\omega}}$ has been redefined. Note that (5.45) is an observability canonical form, with ξ_i, $i \in \omega$, the unobservable parts of the extended state vector. In fact, for a given decentralized control law, (5.36) and (5.45) are just two different realizations of the same transfer-function matrix. However, the form (5.45) leads to the need for a different matrix P_e in (5.1) to guarantee reliable stability and performance, and hence to a different control law. Again, the closed-loop eigenvalues of the controllers which fail appear directly as modes of the closed-loop system; unlike the proof of Theorem 5.3, however, the following development must assume that all the controllers turn out open-loop stable. If some controllers turn out unstable, the design of Theorem 5.4 may be combined with a strongly stabilizing decentralized design developed in Section 5.5.

It is convenient to note that $F_{e\bar{\omega}}$ and $H_{e\bar{\omega}}$ are related to F_e and H_e by

$$F_{e\bar{\omega}} = F_e + \begin{pmatrix} B_\omega \\ 0 \end{pmatrix} (0 \ B_{c\omega}^T X_c) \equiv F_e + B_{e\omega}(0 \ \ B_{c\omega}^T X_c) \tag{5.46a}$$

$$H_{e\bar{\omega}} = H_e + \begin{pmatrix} 0 & 0 \\ 0 & B_{c\omega}^T X_c \end{pmatrix} \tag{5.46b}$$

$$H_{e\omega}^T H_{e\omega} = H_e^T H_e - \begin{pmatrix} 0 \\ X_c B_{c\omega} \end{pmatrix} (0 \ \ B_{c\omega}^T X_c). \tag{5.46c}$$

The following theorem gives the design method:

Theorem 5.4. *With all assumptions and the decentralized design otherwise as in Theorem 3.1, let $X \geq 0$ satisfy*

$$A^T X + XA + \frac{1}{\alpha^2} X G G^T X - X S_\Omega X + H^T H = 0, \tag{5.47}$$

and let $W > 0$ satisfy

$$\begin{aligned} WA_{c+}^T &+ A_{c+}W + \tfrac{1}{\alpha^2}WX_cB_cB_c^TX_cW - WC_c^TC_cW + G_cG_c^T \\ &+ \alpha^2 I_cS_\Omega I_c^T + (W - W_D)C_c^TC_c(W - W_D) = 0, \end{aligned} \tag{5.48}$$

where

$$I_c^T = [I\ I \dots I]$$

$$A_{c+} = A_c + \mathrm{Diag}(S_\Omega X, S_\Omega X, \dots, S_\Omega X),$$

$$S_\Omega = B_\Omega B_\Omega^T,$$

$$S = S_\Omega + S_{\bar{\Omega}},$$

and $\Omega \subseteq \{1, 2, \dots, q\}$. *Let the controllers be given by*

$$\dot{\xi}_i = (A + \frac{1}{\alpha^2}G_+G_+^TX - SX - L_iC_i)\xi_i + L_iy_i, \quad i \in \{1, 2, \dots, q\}, \tag{5.49a}$$

$$u_i = -B_i^TX\xi_i, \quad i \in \{1, 2, \dots, q\}, \tag{5.49b}$$

and assume all controllers are open-loop (internally) stable. Then, for controller outages corresponding to any $\omega \subseteq \Omega$, the closed-loop system (5.45) is internally stable, and the closed-loop transfer-function matrix $T_{\bar{\omega}}(s)$ from $w_{e\bar{\omega}}$ to $z_{\bar{\omega}}$ satisfies $\|T_{\bar{\omega}}\|_\infty \leq \alpha$.

Remark: The design equations (5.47) and (5.48) arise from replacing G in the plant description (1.14) with the augmented matrix G_+ given by

$$G_+ = (G\ \ \alpha B_\Omega), \tag{5.50}$$

and changing the design equations accordingly. This substitution affects both the state-feedback design ARE and the Riccati-like design equation for computing decentralized observer gains. The substitution is equivalent to selecting P_e in (5.1) as

$$P_e = X_e \begin{pmatrix} S_\Omega & 0 \\ 0 & 0 \end{pmatrix} X_e \geq 0. \tag{5.51}$$

The basic design computed for the augmented plant will provide reliable control for the actual plant.

Proof: As in the development of Section 3.1, the existence of appropriate solutions to the design equations (5.47) and (5.48) guarantees that $X_e \geq 0$ satisfies

$$F_e^T X_e + X_e F_e + \frac{1}{\alpha^2} X_e G_{e+} G_{e+}^T X_e + H_e^T H_e = 0. \tag{5.52}$$

Unlike the dual case, the additional columns of G_+ enter into the linear coefficient matrix F_e of (5.52), as well as into the quadratic coefficient as explicitly indicated. This is because the controller structure (5.49) is affected if G is replaced by G_+. Hence, F_e and G_{e+} are now given by

$$F_e = \begin{pmatrix} A & -BB_c^T X_c \\ L_c C & A_{\alpha c} - L_c C_c \end{pmatrix}, G_{e+} = \begin{pmatrix} G_+ & 0 \\ 0 & L_c \end{pmatrix}, \tag{5.53}$$

with $A_{\alpha c} = \mathrm{Diag}(A_\alpha, A_\alpha, \ldots, A_\alpha)$ and $A_\alpha = A + \alpha^{-2} G_+ G_+^T X - SX$. Manipulations of (5.52) similar to those of the dual case, using (5.46), (5.50), and (5.53), give

$$\begin{aligned} F_{e\bar{\omega}}^T X_e \quad &+ \quad X_e F_{e\bar{\omega}} + \tfrac{1}{\alpha^2} X_e G_e G_e^T X_e + H_{e\bar{\omega}}^T H_{e\bar{\omega}} \\ &\leq \quad -(X_e B_{e\omega} - \left(\begin{smallmatrix} 0 \\ X_c B_{c\omega} \end{smallmatrix}\right))(B_{e\omega}^T X_e - (0 \; B_{c\omega}^T X_c) \leq 0. \end{aligned}$$

Provided $(F_{e\bar{\omega}}, H_{e\bar{\omega}})$ is a detectable pair, therefore, Lemma 2.1 guarantees that $F_{e\bar{\omega}}$ is Hurwitz, and that $T_{\bar{\omega}}(s) = H_{e\bar{\omega}}(sI - F_{e\bar{\omega}})^{-1} G_e$ satisfies $\|T_{\bar{\omega}}\|_\infty \leq \alpha$. To establish detectability, let $v^T = (v_1^T \; v_2^T) \neq 0$ satisfy $F_{e\bar{\omega}} v = \lambda v$ and $H_{e\bar{\omega}} v = 0$. Then $Av_1 = \lambda v_1$ and $Hv_1 = 0$. Since (A, H) is a detectable pair, this implies either $Re(\lambda) < 0$ or $v_1 = 0$. Suppose $v_1 = 0$; then $F_{e\bar{\omega}} v = \lambda v$ gives

$$(A_{\alpha c} - L_c C_c) v_2 = \lambda v_2. \tag{5.54}$$

Since all controllers are assumed open-loop stable, (5.54) gives $Re(\lambda) < 0$. Q.E.D.

The two decentralized design methods given in Theorems 5.3 and 5.4 assume controller failures modelled as, respectively, measurement failures and actuator failures. The failures considered incapacitate entire controllers, so that measurement failures and actuator failures have the same effect on the closed-loop transfer-function matrix. Although the two designs have the same reliability goals, they are nevertheless different: The first automatically guarantees reliable stability if the design equations have appropriate solutions, whereas the second may exist but not guarantee reliable stability if the controllers are not open-loop stable; the first design involves only modification of feedback and observer gains as compared with the basic design, while the second requires also a

change in the observer structure; and the range of the design parameter α for which the two designs are computable may differ.

In the centralized case considered in Theorems 5.1 and 5.2, the failures considered are those of individual sensors or actuators. Therefore, the two centralized design methods differ not only in the view taken of controller failure, and in other technical terms, but also in the reliability properties they seek to guarantee.

5.4 Example

For the plant of the example in Section 3.2, the reliable decentralized control design method of Theorem 5.3 was applied for various values of the design parameter α. Table 5.2 gives the actual H_∞ norms of the closed-loop transfer-function matrices resulting when the reliable design was computed for several values of α. For the sake of comparison, the comparable portion of Table 5.1, corresponding to the basic decentralized designs, is reproduced. In addition to H_∞ norms for the case where no controller failure occurs, the H_∞ norms corresponding to a failure of each of the two controllers are given.

Table 5.2 shows that, unlike the basic design, the reliable design guarantees stability and H_∞-norm bound α in spite of a possible failure of Controller #1. In fact, a failure of Controller #1 actually results in a reduced H_∞ norm for the closed-loop system. This is possible because, when Controller #1 fails, the disturbance w_1 and the control input u_1 are removed from consideration. A significant proportion of the cost associated with the design is the result of expended control energy, which is reduced when a controller fails. However, this reduction in cost does not constitute a good argument for discarding Controller #1 and using Controller #2 alone. The use of two controllers increases the reliability of the system, in that a single controller failure will not result in system instability. Note that no solution was found to the Riccati-like algebraic equation for the reliable

Table 5.2. H_∞ norms for basic and reliable decentralized designs.

	Basic Design			Reliable Design		
	no failure	#1 fails	#2 fails	no failure	#1 fails	#2 fails
$\alpha = 20$	3.63	unstable	5.34	6.95	6.25	7.03
$\alpha = 16$	3.63	unstable	5.30	7.65	6.38	7.82
$\alpha = 14$	3.61	unstable	5.28	8.28	6.32	8.59
$\alpha = 12$	3.59	unstable	5.23	No solution to RLAE found.		

design with $\alpha \leq 13$, while solutions were computed for the basic design with the design parameter value as small as $\alpha = 2$. This difference represents the tradeoff between reliability and disturbance attenuation guaranteed by the respective designs.

5.5 Strongly Stabilizing Designs

The designs given in Theorems 5.3 and 5.4 provide decentralized control laws which are reliable with respect to controller outages. For the design given in Theorem 5.3, all controllers susceptible to outages are automatically stable; however, for the design given in Theorem 5.4, the controllers must be assumed to turn out stable for the closed-loop system to be guaranteed stable. A decentralized design is now developed to guarantee open-loop stability of some subset of controllers, without regard for performance in case of a controller outage. This design may be combined with that of Theorem 5.4 so as to guarantee beforehand that specified controllers will turn out open-loop stable. As a special case, a strongly stabilizing centralized design is also derived.

With the design otherwise as in Theorem 3.1, suppose Equation (3.19) is replaced by

$$\begin{aligned} WA_c^T &+ \alpha_C W + \tfrac{1}{\alpha^2} W X_c B_c B_c^T X_c W - W C_c^T C_c W + G_c G_c^T \\ &+ (W - W_D) C_c^T C_c (W - W_D) + P = 0. \end{aligned} \tag{5.55}$$

For any $P \geq 0$, the design guarantees closed-loop stability and the H_∞-norm bound $\|T\|_\infty \leq \alpha$. The object is to select $P \geq 0$ so that the i^{th} controller is open-loop stable. Rewrite (5.55) as

$$W(A_c - L_c C_c)^T + (A_c - L_c C_c)W + \frac{1}{\alpha^2} W X_c B_c B_c^T X_c W + G_c G_c^T + L_c L_c^T + P = 0. \tag{5.56}$$

Recalling the definitions $A_{\alpha c} = \mathrm{Diag}(A_\alpha, A_\alpha, \ldots, A_\alpha)$, $I_c^T = [I\ I \ldots I]$, and $A_c = A_{\alpha c} + I_c B B_c^T X_c$, rewrite (5.56) as

$$\begin{aligned} &W(A_{\alpha c} - L_c C_c)^T + (A_{\alpha c} - L_c C_c)W + \tfrac{1}{\alpha^2} W X_c B_c B_c^T X_c W \\ &+ G_c G_c^T + L_c L_c^T + P + I_c B B_c^T X_c W + W X_c B_c B^T I_c^T = 0. \end{aligned} \tag{5.57}$$

The i^{th} $n \times n$ main-diagonal block of (5.57) is

$$W_{ii}(A_\alpha - L_iC_i)^T + (A_\alpha - L_iC_i)W_{ii} + \frac{1}{\alpha^2}(W_{i1}\dots W_{iq})X_cB_cB_c^TX_c \begin{pmatrix} W_{1i} \\ \vdots \\ W_{qi} \end{pmatrix}$$
$$+GG^T + L_iL_i^T + P_{ii} + BB_c^TX_c \begin{pmatrix} W_{1i} \\ \vdots \\ W_{qi} \end{pmatrix} + (W_{i1}\dots W_{iq})X_cB_cB^T = 0, \tag{5.58}$$

where the linear coefficient $(A_\alpha - L_iC_i)$ is the open-loop dynamic matrix of the i^{th} controller. To ensure that $(A_\alpha - L_iC_i)$ will be Hurwitz, let $P_{ii} = \alpha^2 S = \alpha^2 BB^T$. Then (5.58) becomes

$$W_{ii}(A_\alpha - L_iC_i)^T + (A_\alpha - L_iC_i)W_{ii} + GG^T + L_iL_i^T$$
$$= -(\alpha B + \frac{1}{\alpha}(W_{i1}\dots W_{iq})X_cB_c)(\alpha B^T + \frac{1}{\alpha}B_c^TX_c \begin{pmatrix} W_{1i} \\ \vdots \\ W_{qi} \end{pmatrix}) \leq 0, \tag{5.59}$$

with $W_{ii} > 0$. To see that this is sufficient to guarantee that $(A_\alpha - L_iC_i)$ is Hurwitz, let $v \neq 0$ satisfy $(A_\alpha - L_iC_i)^Tv = \lambda v$. Then (5.59) gives

$$2Re(\lambda)v^*W_{ii}v + v^*L_iL_i^Tv \leq 0,$$

and hence $Re(\lambda) \leq 0$. But inequality must hold here, because $Re(\lambda) = 0$ implies $L_i^Tv = 0$, and hence $A_\alpha^Tv = \lambda v$, with A_α assumed Hurwitz.

Note that $P_{ii} \geq \alpha^2 S$ guarantees that the i^{th} controller will be stable, independent of the other main-diagonal blocks of P. Therefore, several controllers may be simultaneously guaranteed open-loop stable by selecting the main-diagonal blocks P_{ii} of P to satisfy $P_{ii} \geq 0$ if the i^{th} controller need not be stable, and $P_{ii} \geq \alpha^2 S$ if the i^{th} controller must be stable. The other blocks of P may be chosen in any way that makes $P \geq 0$, such as setting them all to 0.

The following theorem summarizes the result.

Theorem 5.5. *Let P be any $qn \times qn$ matrix satisfying*

$$P \geq Diag(P_{11}, \dots, P_{tt}, 0, \dots, 0),$$

where $P_{ii} = \alpha^2 S$ for $i \in \{1,2,\ldots,t\}$. With all assumptions and the design otherwise as in Theorem 3.1, suppose Equation (3.19) is replaced by

$$\begin{aligned} W A_c^T &+ A_c W + \tfrac{1}{\alpha^2} W X_c B_c B_c^T X_c W - W C_c^T C_c W + G_c G_c^T \\ &+ (W - W_D) C_c^T C_c (W - W_D) + P = 0. \end{aligned} \tag{5.60}$$

Then the design, in addition to its other properties, guarantees that the controllers in the first t control channels are all open-loop stable.

The result of Theorem 5.5 is easily specialized to the centralized case. It is important to note, however, that the solution W of the Riccati-like design equation with $q = 1$ is not the same as the solution Y of the observer design ARE in the centralized case. Therefore, the reformulation of the design equations to guarantee strong stabilization in the centralized case is not as simple as that given in Theorem 5.5. The following theorem gives the correct formulation.

Theorem 5.6. *With all assumptions and the design otherwise as in Theorem 2.1, let $Y > 0$ satisfy the ARE*

$$YF^T + FY + \frac{1}{\alpha^2} Y H^T H Y - Y C^T C Y + \frac{1}{\alpha^2} Y X S X Y + G G^T + \alpha^2 S = 0, \tag{5.61}$$

where $F = A - SX$, $S = BB^T$. Then the system is strongly stable, and the closed-loop transfer-function matrix satisfies $\|T\|_\infty \le \alpha$.

Proof: For the special case $q = 1$, the strong stabilization result of Theorem 5.5 still holds. In this case, the design equation (5.60) is

$$\begin{aligned} W(A + \alpha^{-2} G G^T X)^T &+ (A + \alpha^{-2} G G^T X) W + \tfrac{1}{\alpha^2} W X S X W \\ &- W C^T C W + G G^T + \alpha^2 S = 0. \end{aligned} \tag{5.62}$$

Hence, the proof consists of showing that (5.61) implies (5.62). Recall the assumption from Theorem 2.1 that $\sigma_{\max}\{YX\} < \alpha^2$ or $(\alpha^2 Y^{-1} - X) > 0$. This implies that there exists a matrix $W > 0$ such that

$$W^{-1} = Y^{-1} - \alpha^{-2} X. \tag{5.63}$$

Then, routine manipulations of (5.61) give the equivalent equation

$$Y A^T + A Y + \frac{1}{\alpha^2} Y H^T H Y - Y C^T C Y + G G^T + \alpha^2 Y W^{-1} S W^{-1} Y = 0. \tag{5.64}$$

Pre- and post-multiply (5.64) by Y^{-1}, and use (5.63) to obtain

$$\begin{aligned} A^T(\alpha^{-2}X + W^{-1}) \quad &+ \quad (\alpha^{-2}X + W^{-1})A + (\alpha^{-2}X + W^{-1})GG^T(\alpha^{-2}X + W^{-1}) \\ &+ \quad \tfrac{1}{\alpha^2}H^T H - C^T C + \alpha^2 W^{-1}SW^{-1} = 0. \end{aligned} \tag{5.65}$$

Now, divide the state-feedback design ARE (3.3) by α^2 to obtain

$$A^T(\alpha^{-2}X) + (\alpha^{-2}X)A + (\alpha^{-2}X)GG^T(\alpha^{-2}X) - \frac{1}{\alpha^2}XSX + \frac{1}{\alpha^2}H^T H = 0, \tag{5.66}$$

and subtract (5.66) from (5.65) to obtain

$$\begin{aligned} A^T W^{-1} \quad &+ \quad W^{-1}A + (\alpha^{-2}X)GG^T W^{-1} + W^{-1}GG^T(\alpha^{-2}X) \\ &+ \quad W^{-1}GG^T W^{-1} - C^T C + \tfrac{1}{\alpha^2}XSX + \alpha^2 W^{-1}SW^{-1} = 0. \end{aligned} \tag{5.67}$$

Finally, pre- and post-multiply (5.67) by W, and rearrange terms to obtain (5.62). Q.E.D.

Chapter 6

Parameterization of Families of Controllers

6.1 Preliminary Results

This chapter gives some properties of a certain matrix Riccati function related to the computation of families of control laws. The matrix Riccati function is studied in greater detail in [27].

Let F be Hurwitz, and define the matrix Riccati function R by

$$R(X) = F^T X + XF + \frac{1}{\alpha^2} XGG^T X + H^T H.$$

The following property, given in [34], holds:

Lemma 6.1. *If* $R(X) = 0$, *then* $X \geq 0$.

Proof: Suppose $R(X) = 0$; that is,

$$R(X) \equiv F^T X + XF + \frac{1}{\alpha^2} XGG^T X + H^T H = 0. \tag{6.1}$$

Define $P = \frac{1}{\alpha^2} XGG^T X + H^T H$. Then (6.1) becomes $F^T X + XF + P = 0$, with F Hurwitz and $P \geq 0$. By inertia theorems of the Lyapunov equation (see, for example, [16]), $X \geq 0$. Q.E.D.

The following lemma gives a matrix convexity property for R:

Lemma 6.2. *For $i \in \{1,\ldots,r\}$, let X_i be symmetric matrices and β_i be nonnegative scalars satisfying $\sum_{i=1}^r \beta_i = 1$. Then*

$$R\left\{\sum_{i=1}^r \beta_i X_i\right\} \le \sum_{i=1}^r \beta_i R(X_i). \tag{6.2}$$

Proof. Compute

$$\begin{aligned}
R\left\{\sum_{i=1}^r \beta_i X_i\right\} &= F^T\left\{\sum_{i=1}^r \beta_i X_i\right\} + \left\{\sum_{i=1}^r \beta_i X_i\right\} F + \frac{1}{\alpha^2}\left\{\sum_{i=1}^r \beta_i X_1\right\} GG^T \left\{\sum_{i=1}^r \beta_i X_1\right\} + H^T H \\
&= \sum_{i=1}^r (F^T X_i + X_i F + H^T H) + \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^r \beta_i\beta_j X_i GG^T X_j \\
&= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^r X_i GG^T X_i \left\{\sum_{j=1}^r \beta_j\right\} + \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^r \beta_i\beta_j X_i GG^T X_j \\
&= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^r \beta_i\beta_j X_i GG^T (X_i - X_j) \\
&= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^{i-1} \beta_i\beta_j X_i GG^T (X_i - X_j) + \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=i+1}^r \beta_i\beta_j X_i GG^T (X_j - X_i) \\
&= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^{i-1} \beta_i\beta_j X_i GG^T (X_i - X_j) + \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^{i-1} \beta_i\beta_j X_j GG^T (X_i X_j) \\
&= \sum_{i=1}^r \beta_i R(X_i) - \frac{1}{\alpha^2}\sum_{i=1}^r\sum_{j=1}^{i-1} \beta_i\beta_j (X_i - X_j) GG^T (X_i - X_j).
\end{aligned} \tag{6.3}$$

Therefore, $R\{\sum_{i=1}^r \beta_i R(X_i)\} \le \sum_{i=1}^r \beta_i R(X_i)$, the desired result. Q.E.D.

The following corollary identifies a class of easily computable matrices $Z \ge 0$ for which $R(Z) \le 0$:

Lemma 6.3. *Let Z be any convex combination of matrices $X_i \ge 0$, $i \in \{1,2,\ldots,r\}$, satisfying $R(X_i) = 0$. Then $Z \ge 0$ satisfies*

$$R(Z) \le 0. \tag{6.4}$$

Proof. Express Z as

$$Z = \sum_{i=1}^r \beta_i X_i,$$

where $\sum_{i=1}^r \beta_i = 1$. From Lemma 6.1, $Z \ge 0$, and from Lemma 6.2,

$$R(Z) = R\left\{\sum_{i=1}^r \beta_i X_i\right\} \le \sum_{i=1}^r \beta_i R(X_i) = 0.$$

Q.E.D.

6.2 A Family of State-feedback Controls

Consider the plant (1.3) with (A, H) a detectable pair. To derive a characterization of a family of stabilizing state-feedback controls which guarantee a predetermined H_∞-norm bound, start with the design

$$u = -B^T X x \tag{6.5}$$

where

$$A^T X + XA + \frac{1}{\alpha^2} XGG^T X - XBB^T X + H^T H = 0, \quad X \geq 0. \tag{6.6}$$

Rearrange (6.6) as

$$F^T X + XF + \frac{1}{\alpha^2} XGG^T X + (XBB^T X + H^T H) = 0, \tag{6.7}$$

with $F = A - BB^T X$. Since (A, H) is a detectable pair, so is $(F, H^T H + XBB^T X)$. Therefore, by Lemma 2.1, F is Hurwitz, and the transfer function

$$T(s) = \begin{pmatrix} H \\ -B^T X \end{pmatrix} (sI - F)^{-1} G$$

from w_0 to z satisfies $\|T\|_\infty \leq \alpha$. Note that any matrix $X \geq 0$ satisfying (6.6) gives this result, so that if (6.6) has more than one positive semi-definite solution, any one of them can be used to define the control (6.5). In fact, since Lemma 2.1 would still apply if the left-hand side of (6.7) were negative semi-definite, any control law given by

$$u = -B^T Z x, \tag{6.8}$$

$$A^T Z + ZA + \frac{1}{\alpha^2} ZGG^T Z - ZBB^T Z + H^T H \leq 0, \quad Z \geq 0 \tag{6.9}$$

provides stability and the H_∞-norm bound α for the closed-loop system.

A given solution $X \geq 0$ of (6.6) will be called the "central" solution. Given such a central solution, a family of matrices $Z \geq 0$ satisfying (6.9), and hence a family of stabilizing state-feedback control laws which guarantee the closed-loop bound $\|T\|_\infty \leq \alpha$, is characterized.

Take $X \geq 0$ to be the central solution of (6.6), and hence a solution of (6.7) with $F = A - BB^T X$. Given this fixed matrix X, define the matrix Riccati function R by

$$R(M) = F^T M + MF + \frac{1}{\alpha^2} MGG^T M + (XBB^T X + H^T H). \tag{6.10}$$

By Lemma 6.1, each solution of $R(X_i) = 0$ satisfies $X_i \geq 0$. Let $Z \geq 0$ be any convex combination of solutions X_i of $R(X_i) = 0$. By Lemma 6.3,

$$R(Z) \equiv F^T Z + ZF + \frac{1}{\alpha^2} ZGG^T Z + (XBB^T X + H^T H) \leq 0. \tag{6.11}$$

To see that $Z \geq 0$ satisfies (6.9), rearrange (6.11) to obtain

$$\begin{aligned} & A^T Z + ZA + \frac{1}{\alpha^2} ZGG^T Z - ZBB^T Z + H^T H \\ & \quad \leq -ZBB^T Z + ZBB^T X + XBB^T Z - XBB^T X \\ & \quad = -(Z - X)BB^T (Z - X) \leq 0. \end{aligned}$$

The following theorem summarizes the characterization of a family of state-feedback H_∞ controls:

Theorem 6.1. *Let $F = A - BB^T X$ where $X \geq 0$ solves the ARE*

$$A^T X + XA + \frac{1}{\alpha^2} XGG^T X - XBB^T X + H^T H = 0. \tag{6.12}$$

Then, for any convex combination Z of solutions X_i of the ARE

$$F^T X_i + X_i F + \frac{1}{\alpha^2} X_i GG^T X_i + (XBB^T X + H^T H) = 0, \tag{6.13}$$

$F_Z = A - BB^T Z$ is Hurwitz, and the state-feedback control law

$$u = -B^T Z x$$

guarantees that

$$T(s) = \begin{pmatrix} H \\ -B^T Z \end{pmatrix} (sI - F_Z)^{-1} G$$

satisfies $\|T\|_\infty \leq \alpha$.

6.3 A Family of Output-feedback Controls

The approach of Theorem 6.1 extends to the output-feedback case: Start with $Z \geq 0$ a convex combination of solutions X_i of (6.13). Define

$$U_1 = A^T Z + ZA + \frac{1}{\alpha^2} ZGG^T Z - ZBB^T Z + H^T H. \tag{6.14}$$

By Lemma 6.3, $U_1 \leq 0$. The following theorem now gives a family of observers for each state-feedback H_∞ control characterized by such a Z.

Theorem 6.2. *Assume $A + \alpha^{-2} GG^T Z - BB^T Z$ is Hurwitz. Let $Y > 0$ satisfy*

$$AY + YA^T + \frac{1}{\alpha^2} YH^T HY - YC^T CY + GG^T = 0, \tag{6.15}$$

with $(A - YC^T C)$ Hurwitz. Let $V > 0$ be any convex combination of solutions Y_i of

$$(A - YC^T C)Y_i + Y_i(A - YC^T C)^T + \frac{1}{\alpha^2} Y_i(H^T H - U_1)Y_i + (YC^T CY + GG^T) = 0 \tag{6.16}$$

satisfying $\sigma_{\max}\{VZ\} < \alpha^2$, and define the observer gain L by

$$L = (I - \alpha^{-2} VZ)^{-1} VC^T = (V^{-1} - \alpha^{-2} Z)^{-1} C^T. \tag{6.17}$$

Then, the controller

$$\dot{\xi} = (A + \frac{1}{\alpha^2} GG^T Z - BB^T Z - LC)\xi + Ly, \tag{6.18a}$$

$$u = -B^T Zx, \tag{6.18b}$$

stabilizes the plant (1.3), and provides the closed-loop H_∞-norm bound $\|T\|_\infty \leq \alpha$.

Proof. First note that, since $U_1 \leq 0$, $(H^T H - U_1) \geq 0$. By Lemma 6.3,

$$(A - YC^T C)V + V(A - YC^T C)^T + \frac{1}{\alpha^2} V(H^T H - U_1)V + (YC^T CY + GG^T) \leq 0. \tag{6.19}$$

Algebraic manipulations similar to those in the proof of Theorem 6.1 give

$$AV + VA^T + \frac{1}{\alpha^2} VH^T HV - VC^T CV + GG^T \leq \frac{1}{\alpha^2} VU_1 V. \tag{6.20}$$

Pre- and post-multiply (6.20) by αV^{-1} to obtain

$$(\alpha^2 V^{-1})A + A^T(\alpha^2 V^{-1}) + H^T H - \alpha^2 C^T C + \frac{1}{\alpha^2}(\alpha^2 V^{-1})GG^T(\alpha^2 V^{-1}) \leq U_1. \tag{6.21}$$

Subtract (6.14) from (6.21) to obtain

$$\begin{aligned}(\alpha^2V^{-1} - Z)A + A^T(\alpha^2V^{-1} - Z) - \alpha^2C^TC + \frac{1}{\alpha^2}(\alpha^2V^{-1})GG^T(\alpha^2V^{-1}) \\ +ZBB^TZ - \frac{1}{\alpha^2}ZGG^TZ \leq 0\end{aligned} \tag{6.22}$$

Define $X_1 = (\alpha^2V^{-1} - Z) > 0$. Then (6.22) becomes

$$X_1A + A^TX_1 - \alpha^2C^TC + \frac{1}{\alpha^2}(X_1 + Z)GG^T(X_1 + Z) + ZBB^TZ - \frac{1}{\alpha^2}ZGG^TZ \leq 0. \tag{6.23}$$

Now define $U_2 \leq 0$ as the left-hand side of (6.23); rearranging terms, (6.23) becomes

$$\begin{aligned}U_2 \equiv X_1(A + \alpha^{-2}GG^TZ - LC) + (A + \alpha^{-2}GG^TZ - LC)^TX_1 \\ +\alpha^2C^TC + \frac{1}{\alpha^2}X_1GG^TX_1 + ZBB^TZ \leq 0.\end{aligned} \tag{6.24}$$

With the controller (6.18), the closed-loop system transformed to error coordinates is described by

$$\tilde{F}_e = \begin{pmatrix} A - BB^TZ & -BB^TZ \\ \alpha^{-2}GG^TZ & A + \alpha^{-2}GG^TZ - LC \end{pmatrix}, \quad \tilde{G}_e = \begin{pmatrix} G & 0 \\ -G & L \end{pmatrix},$$

$$\tilde{H}_e = \begin{pmatrix} H & 0 \\ -B^TZ & -B^TZ \end{pmatrix}.$$

Define

$$\tilde{X}_e = \begin{pmatrix} Z & 0 \\ 0 & X_1 \end{pmatrix} \geq 0,$$

and consider the quantity

$$\tilde{X}_e\tilde{F}_e + \tilde{F}_e^T\tilde{X}_e + \frac{1}{\alpha^2}\tilde{X}_e\tilde{G}_e\tilde{G}_e^T\tilde{X}_e + \tilde{H}_e^T\tilde{H}_e. \tag{6.25}$$

The two off-diagonal blocks of (6.25) are identically zero. The upper-left block of (6.25) gives U_1 defined in (6.14), and the lower-right block gives U_2 defined in (6.24); therefore,

$$\tilde{X}_e\tilde{F}_e + \tilde{F}_e^T\tilde{X}_e + \frac{1}{\alpha^2}\tilde{X}_e\tilde{G}_e\tilde{G}_e^T\tilde{X}_e + \tilde{H}_e^T\tilde{H}_e = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \leq 0.$$

By Lemma 2.1, $\tilde{F}_e$ is Hurwitz, and $T(s) = \tilde{H}_e(sI - \tilde{F}_e)^{-1}\tilde{G}_e$ satisfies $\|T\|_\infty \leq \alpha$, provided $(\tilde{F}_e, \tilde{H}_e)$ is a detectable pair. Detectability is proved exactly as in the development of Theorem 2.1. Q.E.D.

Recall that Theorem 1.3 gives a parameterization of the set of all output-feedback controllers guaranteeing the H_∞-norm bound α. Some of these controllers are of high order, and are therefore undesirable. By contrast, Theorem 6.2 characterizes a family of controllers with realizations all of the same order as the plant, which all guarantee the H_∞-norm bound α.

6.4 A Family of Decentralized Controls

A generalization of Theorem 6.2 to the decentralized case cannot be readily obtained. Manipulations like those in the proof of Theorem 6.2 applied to the Riccati-like (decentralized) design equation do not give the desired result. Therefore, while Theorem 6.2 gives a family of observer designs for each state-feedback design, the next theorem gives only one decentralized observer design for each state-feedback design of Theorem 6.1. The definitions of Z and U_1 assumed in the theorem statement are as above.

Theorem 6.3. *Assume $A + \alpha^{-2}GG^TZ - BB^TZ$ is Hurwitz and $A + \alpha^{-2}GG^TZ$ has no $j\omega$-axis eigenvalues. Let $W > 0$ satisfy the Riccati-like algebraic equation*

$$\begin{aligned} A_cW + WA_c^T + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW - WC_c^TC_cW \\ +G_cG_c^T + (W - W_D)C_c^TC_c(W - W_D) = 0, \end{aligned} \tag{6.26}$$

and compute $L_c = \mathrm{Diag}\,(L_1, L_2, \ldots, L_q)$ as

$$L_c = W_DC_c^T. \tag{6.27}$$

Then, the control law

$$\dot{\xi}_i = \left(A + \frac{1}{\alpha^2}GG^TZ - BB^TZ - L_iC_i\right)\xi + L_iy_i, \quad i \in \{1, 2, \ldots, q\}, \tag{6.28a}$$

$$u_i = -B_i^TZ\xi_i, \quad i \in \{1, 2, \ldots, q\}, \tag{6.28b}$$

stabilizes the plant (1.14), and provides the closed-loop H_∞-norm bound $\|T\|_\infty \le \alpha$.

Proof: Using (6.27), rewrite (6.26) as

$$(A_c - L_cC_c)W + W(A_c - L_cC_c)^T + \frac{1}{\alpha^2}WX_cB_cB_c^TX_cW + G_cG_c^T + L_cL_c^T = 0. \tag{6.29}$$

Pre- and post-multiply (6.29) by αW^{-1} to obtain

$$\begin{aligned}(\alpha^2W^{-1})(A_c - L_cC_c) + (A_c - L_cC_c)(\alpha^2W^{-1}) + X_cB_cB_c^TX_c \\ +\frac{1}{\alpha^2}(\alpha^2W^{-1})(G_cG_c^T + L_cL_c^T)(\alpha^2W^{-1}) = 0.\end{aligned} \tag{6.30}$$

With controllers (6.28), the closed-loop system is described by the matrices

$$\begin{aligned}\tilde{F}_e &= \begin{pmatrix} A - BB^TZ & -BB_c^TZ_c \\ \alpha^{-2}G_cG^TZ & A_c - L_cC_c \end{pmatrix}, \tilde{G}_e = \begin{pmatrix} G & 0 \\ -G_c & L_c \end{pmatrix}, \\ \tilde{H}_e &= \begin{pmatrix} H & 0 \\ -B^TZ & -B_c^TZ_c \end{pmatrix},\end{aligned} \tag{6.31}$$

where (6.31) differs from (3.11) only in that X has been replaced everywhere by Z. Define

$$\tilde{X}_e = \begin{pmatrix} Z & 0 \\ 0 & \alpha^2W^{-1} \end{pmatrix} \geq 0,$$

and consider the quantity

$$\tilde{X}_e\tilde{F}_e + \tilde{F}_e^T\tilde{X}_e + \frac{1}{\alpha^2}\tilde{X}_e\tilde{G}_e\tilde{G}_e^T\tilde{X}_e + \tilde{H}_e^T\tilde{H}_e. \tag{6.32}$$

The two off-diagonal blocks of (6.32) are identically zero. The upper-left block of (6.32) gives U_1 defined in (6.14). The lower-right block is zero by (6.30). Therefore,

$$\tilde{X}_e\tilde{F}_e + \tilde{F}_e^T\tilde{X}_e + \frac{1}{\alpha^2}\tilde{X}_e\tilde{G}_e\tilde{G}_e^T\tilde{X}_e + \tilde{H}_e^T\tilde{H}_e = \begin{pmatrix} U_1 & 0 \\ 0 & 0 \end{pmatrix} \leq 0.$$

By Lemma 3.1, $(\tilde{F}_e, \tilde{H}_e)$ is a detectable pair; therefore, by Lemma 2.1, the closed-loop system is stable, and the closed-loop transfer-function matrix $T(s) = \tilde{H}_e(sI - \tilde{F}_e)^{-1}\tilde{G}_e$ satisfies $\|T\|_\infty \leq \alpha$. Q.E.D.

Similar to Theorem 6.2 in the centralized case, Theorem 6.3 gives a family of decentralized control laws which guarantee a predetermined H_∞-norm bound for the closed-loop system, and which are characterized by controllers of the same order as the plant. Unlike the centralized case, the family of decentralized controls consists of only a single controller associated with each member of a family of state-feedback controls.

Chapter 7

Conclusions

This thesis has presented a unified approach to the design of robust and reliable decentralized control systems. The basic decentralized design developed includes an observer-type controller in each control channel, which uses a state-feedback model for unknown disturbances. Feedback gains are computed from a state-feedback design ARE, and observer gains are computed from a Riccati-like algebraic equation. The existence of solutions to the design equations guarantees that the closed-loop system matrices satisfy an algebraic Riccati inequality, which is sufficient to establish stability and a predetermined H_∞ disturbance-attenuation bound. This bound is included in the design equations as the parameter α. Appropriate solutions to the design equations will exist only for sufficiently large values of α.

No necessary condition, other than the absence of unstable fixed modes, has been found for the solution of the Riccati-like equation to exist. On the other hand, neither has any sufficient condition for its solution been derived. A simple iterative method for solving the Riccati-like algebraic equation has been used with excellent results; however, there is no guarantee that this iterative method will yield a solution whenever one exists. Existence conditions and computational methods for solutions of the Riccati-like equation are subjects for future research.

A modification of the basic design produces a decentralized control law which is robust with respect to structured plant uncertainty. The design modification is equivalent to including additional disturbances and regulated outputs in the nominal plant description. In addition to the choice of the design parameter α, there is freedom in the robust design to specify bounds σ_k on the norms of uncertainty terms for which robustness is desired. If these bounds are too large,

appropriate solutions to the design equations will not exist. The σ_k's may be varied as design parameters to determine the largest uncertainty bounds for which the robust design exists. Alternatively, the σ_k's may be chosen simply to define a relative weighting to be given to robustness and disturbance-rejection considerations in the design.

Another modification of the basic design produces centralized and decentralized control laws which provide stability and H_∞ disturbance attenuation not only when the system is operating properly, but also in the presence of certain system measurement or control input failures. The design modification is equivalent to including in the plant description additional disturbances or regulated outputs to account for possible control input or measurement outages, respectively. Given the existence of appropriate solutions of the design equations, the reliable designs can tolerate system component outages within a prespecified set of susceptible sensors or actuators in the centralized case, or within a prespecified set of susceptible controllers in the decentralized case. Of course, for appropriate solutions of the design equations to exist, the measurement and control components not included in the susceptible set must be able to stabilize the system by themselves.

In the case of control laws designed to tolerate possible actuator outages, the additional condition that the control law be open-loop stable is required. If such a design is attempted, but results in an unstable controller, closed-loop system stability is not guaranteed. In this case, the design can be further modified to include strong-stabilization properties. In the decentralized case, for example, the combined design modifications would consist of (i) appending the columns of αB_Ω to the disturbance matrix G in the plant description, and (ii) adding a constant block-diagonal matrix, with $\alpha^2 BB^T$ blocks on its main diagonal, to the left-hand side of the Riccati-like design equation.

The robust and reliable designs are obtained at the cost of allowing a higher H_∞ disturbance-attenuation bound. This is natural, since the larger the H_∞-norm bound is allowed to be, the larger is the set of controllers which will guarantee that bound. For a sufficiently large bound, then, the corresponding set of controllers may include controllers with desired special properties. The design methods presented in this thesis select such controllers from among all controllers which guarantee the specified disturbance-attenuation bound.

A convexity property of a certain matrix Riccati function is used to develop a parameterization of families of controllers which provide stability and H_∞ disturbance attenuation. This param-

eterization has two advantages over the one given in Theorem 1.3: First, it includes families of decentralized control laws. Second, it gives only controller realizations of the same order as the plant. Hence, the order of the plant is an upper bound for the order of a minimal realization of any of the controllers in a given family. One criterion for choosing among the controllers could be the order of their minimal realizations. How to choose from the family a controller with a lower-order minimal realization is a problem for future research.

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